Super Element

A super element is a complex type of element, which includes a number of finite elements used in a structural modeling. It is also termed as a substructure. The substructure is a well-known technique used frequently in structural engineering to reduce the size of the problem to be solved. In MIDAS Programs, however, the super element procedure was developed in order to enhance the modeling capabilities.

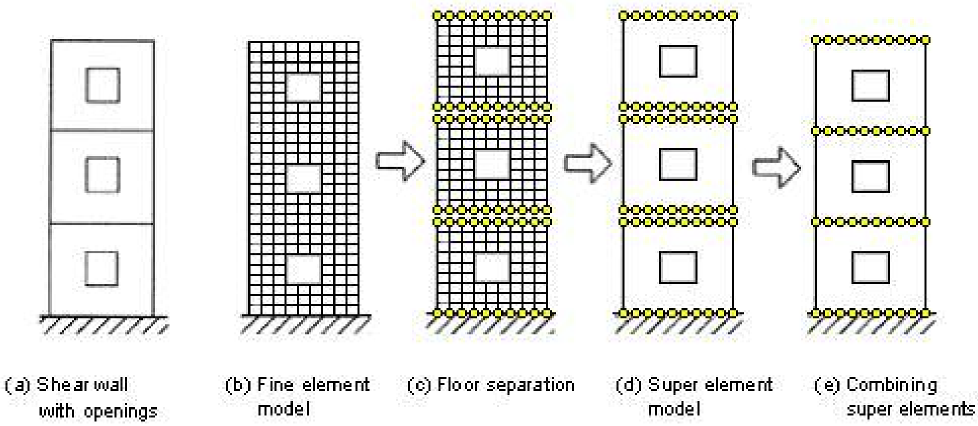

Figure *Modeling process of shear walls with openings using super elements*

When a building structure model consists of more than beams and columns but includes also shear-walls and slabs, so called "connectivity difficulties" are encountered at the nodes where linear and planar elements are connected. The super element procedure is used to resolve these difficulties. Application of super elements in modeling process a shear walls with openings is shown schematically on the Figure above. The structural elements of the shear wall are finely meshed, and the degrees of freedom pertaining to all the nodes except for the nodes connected to beams and columns are eliminated through the process of static condensation technique. This super element is employed in the RDS (Residential Design System) program being developed.

The formulation of a super element, which is used to model shear walls and slabs, is outlined below. In order to construct a super element, the stiffness of an assemblage of plane elements represented by refined meshes must be constructed. Let us define the stiffness matrices and loads of the assemblage of elements having n meshes as $\mathbf{k}_1$, $\mathbf{k}_2$, …, $\mathbf{k}_n$ and $\mathbf{f}_1$, $\mathbf{f}_2$, …, $\mathbf{f}_n$, respectively. These matrices, as in the general finite element analysis, should be assembled for each DOF. Thus we define the assembly stiffness matrix and the load vector as $\mathbf{K}$ and $\mathbf{f}$ respectively.

In the matrix $\mathbf{K}$ and vector $\mathbf{f}$, we can classify the terms related to the DOF that will be eliminated through the process of condensation and the terms related to the DOF that will be maintained, which are expressed as follows:

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{uc} \\ \mathbf{K}_{cu} & \mathbf{K}_{cc} \end{bmatrix} \begin{bmatrix} \mathbf{u}_u \\ \mathbf{u}_c \end{bmatrix} = \begin{bmatrix} \mathbf{f}_u \\ \mathbf{f}_c \end{bmatrix}$$

where, index u denotes the terms related to the DOF of super nodes that will remain after condensation, and c represents the terms related to the DOF internal nodes that will disappear through condensation.

Now, introducing the matrix condensation procedure, we can obtain,

$$\mathbf{u}_c = \mathbf{K}_{cc}^{-1}\left(\mathbf{f}_c - \mathbf{K}_{cu}\mathbf{u}_u\right)$$
$$\left(\mathbf{K}_{uu} - \mathbf{K}_{uc}\mathbf{K}_{cc}^{-1}\mathbf{K}_{cu}\right)\mathbf{u}_u = \mathbf{f}_u - \mathbf{K}_{uc}\mathbf{K}_{cc}^{-1}\mathbf{f}_c$$

Accordingly, the new stiffness matrix and load vector of a super element are given by,

$$\overline{\mathbf{K}} = \mathbf{K}_{uu} - \mathbf{K}_{uc}\mathbf{K}_{cc}^{-1}\mathbf{K}_{cu}$$
$$\overline{\mathbf{f}} = \mathbf{f}_u - \mathbf{K}_{uc}\mathbf{K}_{cc}^{-1}\mathbf{f}_c$$

The above describes the super element procedure employed in static analysis. In dynamic analysis on the other hand, the dynamic condensation procedure is required. The dynamic equilibrium equation neglecting the damping effects is given by,

$$\begin{bmatrix} \mathbf{M}_{uu} & \mathbf{M}_{uc} \\ \mathbf{M}_{cu} & \mathbf{M}_{cc} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_u \\ \ddot{\mathbf{u}}_c \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{uc} \\ \mathbf{K}_{cu} & \mathbf{K}_{cc} \end{bmatrix} \begin{bmatrix} \mathbf{u}_u \\ \mathbf{u}_c \end{bmatrix} = \begin{bmatrix} \mathbf{f}_u \\ \mathbf{f}_c \end{bmatrix}$$

Now, if we assume that the displacement terms of the DOF to be eliminated are obtained by static condensation in the case where the loads acting on the corresponding terms are 0, the following equation is established:

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{uc} \\ \mathbf{K}_{cu} & \mathbf{K}_{cc} \end{bmatrix} \begin{bmatrix} \mathbf{u}_u \\ \mathbf{u}_c \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$
$$\therefore \mathbf{u}_c = -\mathbf{K}_{cc}^{-1}\mathbf{K}_{cu}\mathbf{u}_u = \mathbf{T}_{cu}\mathbf{u}_u$$

The accelerations are obtained by the second order differentiation of displacements with time.

$$\ddot{\mathbf{u}}_c = \mathbf{T}_{cu}\ddot{\mathbf{u}}_u$$

Then, the transformation matrix below is introduced.

$$\mathbf{T}_u = \begin{bmatrix} \mathbf{I}_u \\ \mathbf{T}_{cu} \end{bmatrix}$$

(where, $\mathbf{I}_u$ is an identity matrix which retains the same order as $\mathbf{K}_{uu}$)

Using this transformation matrix and its transpose, we can obtain the following equation:

$$\mathbf{T}_u^T \begin{bmatrix} \mathbf{M}_{uu} & \mathbf{M}_{uc} \\ \mathbf{M}_{cu} & \mathbf{M}_{cc} \end{bmatrix} \mathbf{T}_u \ddot{\mathbf{u}}_u + \mathbf{T}_u^T \begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{uc} \\ \mathbf{K}_{cu} & \mathbf{K}_{cc} \end{bmatrix} \mathbf{T}_u \mathbf{u}_u = \mathbf{T}_u^T \begin{bmatrix} \mathbf{f}_u \\ \mathbf{f}_c \end{bmatrix}$$

$$\therefore \overline{\mathbf{M}} \ddot{\mathbf{u}}_u + \overline{\mathbf{K}} \mathbf{u}_u = \overline{\mathbf{f}}$$

where,

$$\overline{\mathbf{M}} = \mathbf{T}_u^T \mathbf{M} \mathbf{T}_u = \mathbf{M}_{uu} + \mathbf{T}_{cu}^T \mathbf{M}_{cu} + \mathbf{M}_{uc} \mathbf{T}_{cu} + \mathbf{T}_{cu}^T \mathbf{M}_{cc} \mathbf{T}_{cu}$$

and $\overline{\mathbf{K}}, \overline{\mathbf{f}}$ are identical to the terms specified in static condensation procedure.

When the dynamic analysis is performed with damping effects included, than a proportional damping matrix should be obtained by the same procedure, and $\overline{\mathbf{C}} = \mathbf{T}_u^T \mathbf{C} \mathbf{T}_u$.