Solid Elements

The Finite Element Library of the MIDAS Family Programs includes the following Solid Elements: 4-node tetrahedron, 6-node pentahedron, and 8-node hexahedron shown in Fig. 1. The finite element formulation of all element types is based on the *isoparametric* procedure (the element geometry and displacements are interpolated in the same way).

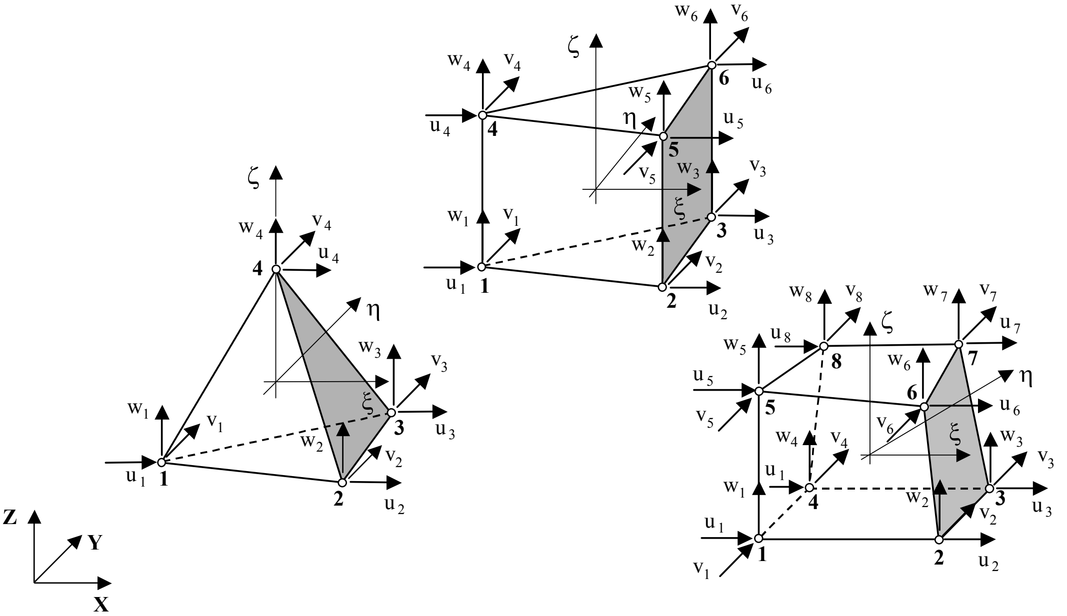


Figure 1 *3D Solid Finite Elements*

The nodal degrees of freedom (DOF's) are illustrated in Fig. 1. The element geometry and displacement field are defined in terms of nodal coordinated and DOF's by the following functions:

$$
\begin{cases}
x = \sum\limits_{i=1}^{n} f_i(\xi,\eta,\zeta)\, x_i \\
y = \sum\limits_{i=1}^{n} f_i(\xi,\eta,\zeta)\, y_i \\
z = \sum\limits_{i=1}^{n} f_i(\xi,\eta,\zeta)\, z_i
\end{cases}
\qquad
\begin{cases}
u = \sum\limits_{i=1}^{n} f_i(\xi,\eta,\zeta)\, u_i \\
v = \sum\limits_{i=1}^{n} f_i(\xi,\eta,\zeta)\, v_i \\
w = \sum\limits_{i=1}^{n} f_i(\xi,\eta,\zeta)\, w_i
\end{cases}
\Rightarrow \mathbf{u} = \mathbf{fq}
$$

where

$\mathrm{u}_i, \mathrm{v}_i, \mathrm{w}_i$ = displacements in direction of global $\mathrm{X}, \mathrm{Y}, \mathrm{Z}-$axes, respectively, at $i-$th node

x_i, y_i, z_i = $x, y, z-$coordinates at $i-$th node

$f_i(\xi,\eta,\zeta)$ = interpolation function related to $i-$th node and defined in the natural coordinate system of the element with variables ξ,η,ζ ($-1 \le \xi \le +1, -1 \le \eta \le +1$, and $-1 \le \zeta \le +1$)

ξ_i, η_i, ζ_i = natural coordinates of the $i-$th node of the element

n = number of nodes in element

The nodal interpolation functions $f_i(\xi,\eta,\zeta)$ are of the following form:

For 4-node tetrahedron element

$$f_i(\xi,\eta,\zeta) = \begin{cases} \frac{1}{2}(1+\zeta) & \text{for } i = 4 \\ \frac{1}{4}(1+\xi)(1-\zeta) & \text{for } i = 3 \\ \frac{1}{8}(1+\xi_i\xi)(1-\eta)(1-\zeta) & \text{for } i = 1,2 \end{cases}$$

For 6-node pentahedron (wedge) element

$$f_i(\xi,\eta,\zeta) = \begin{cases} \frac{1}{4}(1+\xi)(1+\zeta_i\zeta) & \text{for } i = 3,6 \\ \frac{1}{8}(1+\xi_i\xi)(1+\eta_i\eta)(1+\zeta_i\zeta) & \text{for } i = 1,2,4,5 \end{cases}$$

For 8-node hexahedron (brick) element

$$f_i(\xi,\eta,\zeta) = \frac{1}{8}(1+\xi_i\xi)(1+\eta_i\eta)(1+\zeta_i\zeta) \quad \text{for } i = 1,2,3,.....,8$$

Now, the strains at any point within element domain are expressed in terms of nodal displacements as,

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{bmatrix}=\begin{bmatrix}\dfrac{\partial u}{\partial x}\\ \dfrac{\partial v}{\partial y}\\ \dfrac{\partial w}{\partial z}\\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\\ \dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y}\\ \dfrac{\partial w}{\partial x}+\dfrac{\partial u}{\partial z}\end{bmatrix}=\begin{bmatrix}\dfrac{\partial}{\partial x} & 0 & 0\\ 0 & \dfrac{\partial}{\partial y} & 0\\ 0 & 0 & \dfrac{\partial}{\partial z}\\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0\\ 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y}\\ \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial x}\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix}\Rightarrow \varepsilon=\mathbf{du}=\mathbf{dfq}=\mathbf{Bq}$$

where

B = strain-displacement matrix

q = vector of nodal displacements

Then, the stress-strain relation become,

$$\sigma=\mathbf{D}\varepsilon=\mathbf{DBq}$$

where, **D** is the elasticity matrix defining mechanical properties of the material. For a linear isotropic material **D** matrix takes the following form:

$$\mathbf{D}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ & 1-\nu & \nu & 0 & 0 & 0\\ & & 1-\nu & 0 & 0 & 0\\ & & & \dfrac{1-2\nu}{2} & 0 & 0\\ & \text{Sym.} & & & \dfrac{1-2\nu}{2} & 0\\ & & & & & \dfrac{1-2\nu}{2}\end{bmatrix}$$

in which

E = Young modulus

ν = Poisson’s ratio

Accordingly, the stiffness matrix and force vectors for a typical isoparametric 3D solid element are defined by the following integrals:

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} \, dV$$

$$\mathbf{p}_b = \int_\Omega \mathbf{f}^T \mathbf{b} \, d\Omega$$

$$\mathbf{p}_0 = \int_\Omega \mathbf{B}^T \mathbf{D} \underline{\varepsilon_0} \, d\Omega$$

where

$\mathbf{K}$ = stiffness matrix

V = volume of the element

A = surface area of the element

$\mathbf{p}_b$ = nodal force vector due to distributed body forces $\mathbf{b}$

$\mathbf{p}_0$ = nodal force vector due to initial strain ε_0

Ω = range of the integration

So far we have demonstrated the standard *isoparametric* formulation procedure, which is identical for all 3D solids elements. It should be noted the 4-node tetrahedron and 6-node wedge elements presented here are in fact the degenerated forms of the 8-node hexahedron (brick). Formulation of these elements by collapsing the 8-node hexahedra allows evaluating the above integrals for all types of 3D solid elements by use of the same standard 3D numerical integration procedure, based on the Gauss-Legendre quadrature. Thus the numerical integration formulas used for 3D elements are:

$$\mathbf{K} = \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} W_i W_j W_k \mathbf{B}^T \left(\xi_i, \eta_k, \zeta_k\right) \mathbf{D} \mathbf{B} \left(\xi_i, \eta_k, \varsigma_k\right) \left| \mathbf{J} \left(\xi_i, \eta_k, \varsigma_k\right) \right|$$

$$\mathbf{p}_b = \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} W_i W_j W_k \mathbf{f}^T \left(\xi_i, \eta_k, \zeta_k\right) \mathbf{b} \left(\xi_i, \eta_k, \zeta_k\right) \left| \mathbf{J} \left(\xi_i, \eta_k, \zeta_k\right) \right|$$

$$\mathbf{p}_0 = \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} W_i W_j W_k \mathbf{B}^T \left(\xi_i, \eta_k, \zeta_k\right) \mathbf{D} \varepsilon_0 \left(\xi_i, \eta_k, \varsigma_k\right) \left| \mathbf{J} \left(\xi_i, \eta_k, \zeta_k\right) \right|$$

where:

W_i = weighting factor of i-th integration point

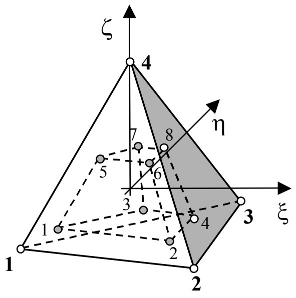
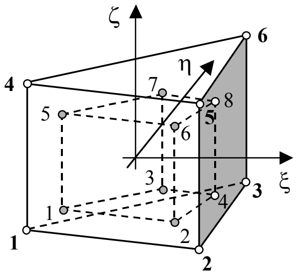
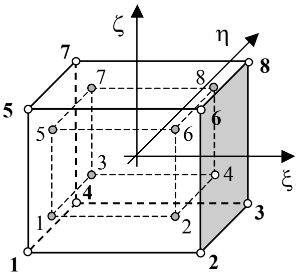
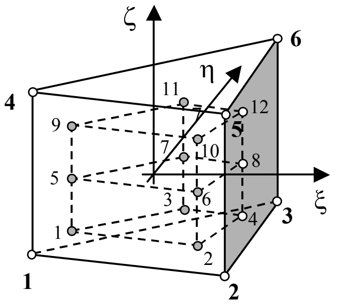
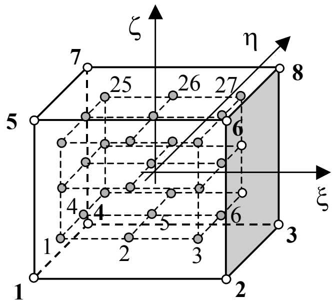
ξ_i, η_i, ζ_i = natural coordinates of i-th integration point

$\left| \mathbf{J} \left(\xi_i, \eta_j, \zeta_k\right) \right|$ = determinant of the *Jacobian* matrix

l, m, n = number of integration points in direction of ξ, η, and ζ, respectively.

The appropriate order of numerical integration and corresponding locations of integration points used in the 3D solid elements are shown in Table bellow.

Table *Gauss-Legendre Integration for 3D Solid Elements*

Integration order	**Location of integration points**
2×2×2	  
3×3×3	 

Here it should be noted that the compatible 8-node hexahedron element based on standard *isoparametric* formulation does not produce accurate results in many cases for both displacements and stresses. In order to improve the performance of this element are added so called the *incompatible displacements modes*. Taking into account the incompatible modes, the displacement field is defined as,

$$\begin{cases} u = \sum_{i=1}^{8} f_i\left(\xi,\eta,\zeta\right)_i u_i + \alpha_1 P_1 + \alpha_4 P_2 + \alpha_7 P_3 \\ v = \sum_{i=1}^{8} f_i\left(\xi,\eta,\zeta\right)_i v_i + \alpha_2 P_1 + \alpha_5 P_2 + \alpha_8 P_3 \\ w = \sum_{i=1}^{8} f_i\left(\xi,\eta,\zeta\right) w_i + \alpha_3 P_1 + \alpha_6 P_2 + \alpha_9 P_3 \end{cases}$$

where, $P_1,\ P_2, P_3$ are the extra shape functions related to the incompatible modes expressed as,

$$P_1\left(\xi,\eta,\zeta\right) = 1-\xi^2,\ P_2\left(\xi,\eta,\zeta\right) = 1-\eta^2 \text{ and } P_3\left(\xi,\eta,\zeta\right) = 1-\zeta^2$$

and $\alpha_i\ (i = 1,2,3,...,9)$ are so called "nodeless extra DOF's". Note that the extra shape functions permit a parabolic deformation along the element edge and improve the element stiffness performance.

Here it should be noted that the final stiffness matrix is obtained by static condensation of the incompatible modes in the same way as in case of 4-node quadrilateral 2D plane element.