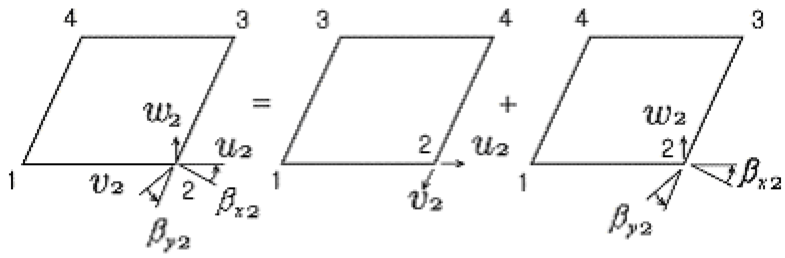
Plate (Shell) Elements

The Finite Element Library of the MIDAS Family Programs includes 3-node triangular and 4-node quadrilateral Plate/Shell elements.

The MIDAS shell elements are flat shell elements and are formulated as a combination of a plane stress and plate-bending elements. The stiffness terms associated with two in-plane translational degrees of freedom (DOF) are formulated in the same way as for plane stress element. The stiffness terms related to one out-of-plane translational DOF and two out-of-plane rotational DOF are formulated by the same procedure used for the plate-bending element.



Shell Plane stress Plate-bending

Figure 1 *Formulation of a Shell Element*

For details on formulation of a plane stress element, refer to “Plane Stress & Plane Strain Element”.

A plate-bending element is used to model a shell subjected to out-of-plane bending. This element is classified into a thin plate element, which ignores shear deformations and a thick plate element, which accounts for the effects of shear deformations.

The formulation of a thin plate element is based on the concept of Discrete Kirchhoff Triangular (DKT) element. As shown in Fig. 2., this element has one translational DOF and two rotational DOF in the local coordinates. A four node quadrilateral element is constructed using this DKT element. As shown below, a quadrilateral is divided into two triangles, and the two triangular elements are superimposed. Since there are two directions of diagonals, the four triangular elements are superimposed, and the resultant stiffness is obtained by dividing the sum by 2 (see Fig. 2).

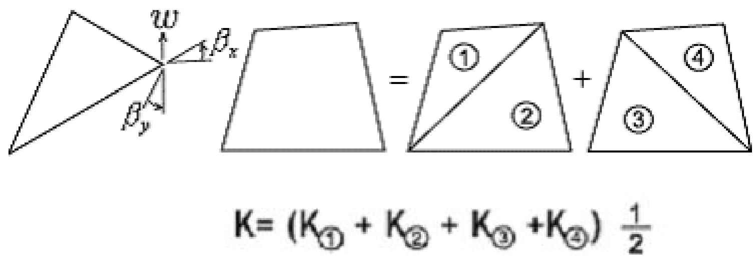


Figure 2 *Formulation of a Plate Element based on the DKT Element*

The plate finite element formulation based on the concept of the DKT element is described below. This element is derived from the assumptions ignoring shear deformations (Kirchhoff thin plate hypothesis). The plate theory accounting for transverse shear deformations (Mindlin-Reissner plate theory) is based on the following assumption: "The particles located normal to the neutral surface of an undeformed plate, after deformation, remain on a straight line but this straight line is not necessarily normal to the deformed neutral surface." Using this assumption, the displacement components at an arbitrary point in the element are,

$$u = z\beta_x(x,y),\ v = z\beta_y(x,y),\ w = w(x,y)$$

where, β_x and β_y are the rotation angles of the lines normal to the undeformed neutral surface in the **x-z** and **y-z** planes, respectively. According to Kirchhoff plate theory, which ignores shear deformations,

$$\beta_x = -\frac{\partial w}{\partial x} \text{ and } \beta_y = \frac{\partial w}{\partial y}$$

The equation for linear flexural strains is,

$$\underline{\varepsilon}_b = z\underline{\kappa} = z\begin{bmatrix} \dfrac{\partial \beta_x}{\partial x} \\ \dfrac{\partial \beta_y}{\partial y} \\ \dfrac{\partial \beta_x}{\partial y} + \dfrac{\partial \beta_y}{\partial x} \end{bmatrix}$$

where, z is the distance from the neutral surface and $\underline{\kappa}$ is the vector of curvatures.

And the equation for shear strains is,

$$\underline{\gamma} = \begin{bmatrix} \dfrac{\partial w}{\partial x} + \beta_x \\ \dfrac{\partial w}{\partial y} + \beta_y \end{bmatrix}$$

The state of the stresses is assumed equal to that of a plane stress element. Therefore, for an isotropic element, the flexural stresses of the element are,

$$\underline{\sigma}_b = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = z\mathbf{D}\underline{\kappa} = z\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\underline{\kappa}$$

and the shear strains are,

$$\underline{\sigma}_s = \begin{bmatrix} \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \mathbf{E}\underline{\gamma} = \frac{E}{2(1+\nu)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\underline{\gamma}$$

Now, the strain energy is obtained as follows:

$$U = U_b + U_s = \frac{1}{2}\int_A \underline{\kappa}^T \mathbf{D}_b \underline{\kappa} dA + \frac{1}{2}\int_A \overline{\gamma}^T \mathbf{D}_s \gamma dA$$

where,

D$_b$ = Stress-strain (material) matrix for bending

D$_s$ = Stress-Strain matrix for shear

For an isotropic material, **D**$_b$ and **D**$_s$ take the following form:

$$\mathbf{D}_b = \int_{-h/2}^{h/2} \mathbf{D} z^2 dz = \frac{Eh^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

$$\mathbf{D}_s = k\int_{-h/2}^{h/2} \mathbf{E} dz = \frac{kEh}{2(1+\nu)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

in which, k is the correction factor for considering the non-uniform characteristic of shear strains, and is usually assumed as 5/6.

Then, by definition, flexural moments **M** and shear forces **Q** are obtained through integration of stresses over the thickness as follows:

$$\mathbf{M} = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \sigma z dz = \mathbf{D}_b \underline{\kappa}$$

$$\mathbf{Q} = \begin{bmatrix} Q_x \\ Q_y \end{bmatrix} = k\int_{-h/2}^{h/2} \sigma_s dz = \mathbf{D}_s \underline{\gamma}$$

The boundary condition of this element is $\mathbf{C}^0$ continuation condition, which includes only the compatibility of displacements and rotations. Since it does not involve differential terms, it is an easy task to satisfy the condition.

The above equations represent the formulation of a thick plate element, which includes shear deformations. Now, we proceed to formulation of a DKT element without shear deformations. The DKT element and node numbering convention is illustrated in Fig. 3. We first ignore the $\boldsymbol{U_s}$ term and assume the following:

(1) The element has only three DOF, i.e., displacement w and two rotations θ_x and θ_y at each node.

(2) Since the shear deformations are ignored, the nodal rotations must satisfy the Kirchhoff boundary conditions, $\theta_x = \dfrac{\partial w}{\partial y}$ and $\theta_y = -\dfrac{\partial w}{\partial x}$.

(3) The assumptions of the Kirchhoff plate theory can be constrained at some discrete points.

(4) The compatibility condition of rotations, β_x and β_y, must be satisfied.

Also, we assume the following to formulate a DKT element:

(1) The rotations β_x and β_y are imposed in a quadratic variation in the element, i.e.,

$$\beta_x = \sum_{i=1}^{6} f_i \beta_{xi} \text{ and } \beta_y = \sum_{i=1}^{6} f_i \beta_{yi} .$$

where, β_{xi} and β_{yi} represent the values of rotations at each node and mid-edge point, f_i represents the shape functions associated with i-th node and defined as follows:

$$\begin{cases} f_1 = \xi_1 (2\xi_1 - 1) \\ f_2 = \xi_2 (2\xi_2 - 1) \\ f_3 = \xi_3 (2\xi_3 - 1) \\ f_4 = 4\xi_2 \xi_3 \\ f_5 = 4\xi_3 \xi_1 \\ f_6 = 4\xi_1 \xi_2 \end{cases} \qquad \xi_1 + \xi_2 + \xi_3 = 1$$

(2) The Kirchhoff hypothesis is constrained at each node and mid-node. That is, the following conditions must be satisfied:

$$\underline{\gamma} = \begin{bmatrix} \beta_x + \dfrac{\partial w}{\partial x} \\ \beta_y + \dfrac{\partial w}{\partial y} \end{bmatrix} = \mathbf{0} \quad \text{(at nodes 1, 2 \& 3)}$$

$$\beta_s + \frac{\partial w}{\partial s} = 0 \quad \text{(at mid-edge points)}$$

where, s is the direction along the edge.

(3) The variation of w along the edge is cubic, i.e.,

$$\left(\frac{\partial w}{\partial s}\right)_k = -\frac{3}{2l_{ij}} w_i - \frac{1}{4}\left(\frac{\partial w}{\partial s}\right)_i + \frac{3}{2l_{ij}} w_j - \frac{1}{4}\left(\frac{\partial w}{\partial s}\right)_j$$

where, k denotes the mid-node of the side ij, and l_{ij} represents the length of the side ij.

(4) The rotational angle β_n about the tangential direction to the side is assumed to vary linearly along the side, i.e.,

$$\beta_{nk} = \frac{1}{2}\left(\beta_{ni} + \beta_{nj}\right)$$

Now, the nodal degrees of freedom are,

$$\mathbf{u} = \begin{bmatrix} w_1 & \theta_{x1} & \theta_{y1} & w_2 & \theta_{x2} & \theta_{y2} & w_3 & \theta_{x3} & \theta_{y3} \end{bmatrix}^T$$

The following geometrical relations are required on each side:

$$\begin{bmatrix} \beta_x \\ \beta_y \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} \beta_n \\ \beta_s \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial w}{\partial s} \\ \dfrac{\partial w}{\partial n} \end{bmatrix} = \begin{bmatrix} c & s \\ s & -c \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix}$$

where, $c = \cos(x, n_{ij})$ and $s = \sin(x, n_{ij})$

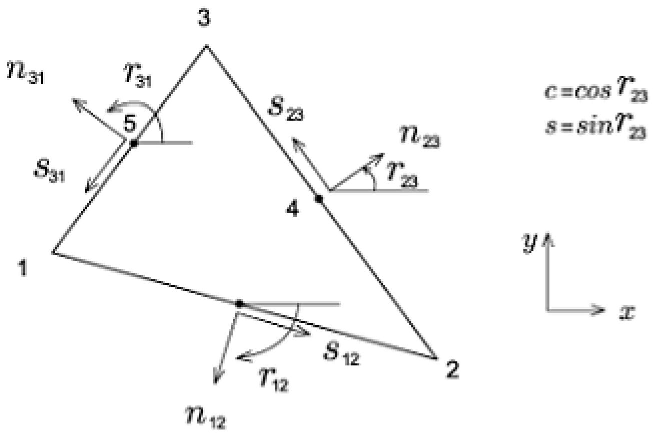


Figure 3 *Node numbering convention for DKT element*

β_x and β_y can be obtained by the assumed shape functions as follows:

$$\beta_x = \mathbf{H}_x^T(\xi_2,\xi_3)\mathbf{U} = \begin{bmatrix} 1.5(a_6 f_6 - a_5 f_5) \\ b_5 f_5 + b_6 f_6 \\ f_1 - c_5 f_5 - c_6 f_6 \\ 1.5(a_4 f_4 - a_6 f_6) \\ b_6 f_6 + b_4 f_4 \\ f_1 - c_6 f_6 - c_4 f_4 \\ 1.5(a_5 f_5 - a_4 f_4) \\ b_4 f_4 + b_5 f_5 \\ f_1 - c_4 f_4 - c_5 f_5 \end{bmatrix}^T \begin{bmatrix} w_1 \\ \theta_{x_1} \\ \theta_{y_1} \\ w_2 \\ \theta_{x_2} \\ \theta_{y_2} \\ w_3 \\ \theta_{x_3} \\ \theta_{y_3} \end{bmatrix}$$

$$\beta_y = \mathbf{H}_y^T(\xi_2,\xi_3)\mathbf{U} = \begin{bmatrix} 1.5(d_6 f_6 - d_5 f_5) \\ -f_1 + e_5 f_5 + e_6 f_6 \\ -(b_5 f_5 + b_6 f_6) \\ 1.5(d_4 f_4 - d_6 f_6) \\ -f_1 + e_6 f_6 + e_4 f_4 \\ -(b_6 f_6 + b_4 f_4) \\ 1.5(d_5 f_5 - d_4 f_4) \\ -f_1 + e_4 f_4 + e_5 f_5 \\ -(b_4 f_4 + b_5 f_5) \end{bmatrix}^T \begin{bmatrix} w_1 \\ \theta_{x_1} \\ \theta_{y_1} \\ w_2 \\ \theta_{x_2} \\ \theta_{y_2} \\ w_3 \\ \theta_{x_3} \\ \theta_{y_3} \end{bmatrix}$$

where

$$a_k = -x_{ij}/l_{ij}^2,\ b_k = \frac{3}{4}x_{ij}y_{ij}/l_{ij}^2,\ c_k = \left(\frac{1}{4}x_{ij}^2 - \frac{1}{2}y_{ij}^2\right)\Big/ l_{ij}^2$$

$$d_k = -y_{ij}/l_{ij}^2,\ e_k = \left(\frac{1}{4}y_{ij}^2 - \frac{1}{2}x_{ij}^2\right)\Big/ l_{ij}^2,\ l_{ij}^2 = x_{ij}^2 + y_{ij}^2$$

in which for ij = 23, 31 & 12, k = 4, 5 & 6, respectively, and $x_{ij} = x_i - x_j$, $y_{ij} = y_i - y_j$

Then, the vector of curvatures $\underline{\kappa}$ is obtained as follows:

$$\underline{\kappa} = \mathbf{Bu}$$

$$\mathbf{B}(\xi_2,\xi_3) = \frac{1}{2A}\begin{bmatrix} y_{31}\mathbf{H}_{x,\xi_2}^T + y_{12}\mathbf{H}_{x,\xi_3}^T \\ -x_{31}\mathbf{H}_{y,\xi_2}^T - x_{12}\mathbf{H}_{y,\xi_3}^T \\ -x_{31}\mathbf{H}_{x,\xi_2}^T - x_{12}\mathbf{H}_{x,\xi_3}^T + y_{31}\mathbf{H}_{y,\xi_2}^T + y_{12}\mathbf{H}_{y,\xi_3}^T \end{bmatrix}$$

where, A is the area of the element

Accordingly, the stiffness matrix of the DKT element becomes,

$$\mathbf{K}_{DKT} = 2A\int_0^1\int_0^{1-\xi_3} \mathbf{B}^T \mathbf{D}_b \mathbf{B} d\xi_2 d\xi_3$$

The stiffness matrix in the above equation is calculated by numerical integration using the 3-point Gaussian quadrature.

Also, the bending moments $\mathbf{M}$ at any point in the element are obtained as follows:

$$\mathbf{M}(x,y) = \mathbf{D}_b \mathbf{B}(x,y)\mathbf{U}$$

$$\begin{cases} x = x_1 + \xi_2 x_{21} + \xi_3 x_{31} \\ y = y_1 + \xi_2 y_{21} + \xi_3 y_{31} \end{cases}$$

The thick plate element included in MIDAS Finite Element Library is a modified Discrete Kirchhoff-Mindlin Triangular/Quadrilateral (DKMT/DKMQ) element.

The formulation procedure for the DKMT element is quite similar to that used for DKT element. Except, the terms related to the shear stiffness are included in the formulation. Also, for the sake of convenience of formulation, some parts are expressed in somewhat different forms.

First, the equation for the curvatures and the stress-strain relation are identical to those used in the DKT element.

The DKMT element and node numbering convention is illustrated in Fig. 4. The shape functions are expressed by dividing the node terms and mid-node terms as follows:

$$\begin{cases} \beta_x = \sum_{i=1}^{3} f_i \beta_{xi} + \sum_{k=4}^{6} P_k c_k \Delta\beta_{sk} \\ \beta_y = \sum_{i=1}^{3} f_i \beta_{yi} + \sum_{k=4}^{6} P_k s_k \Delta\beta_{sk} \end{cases}$$

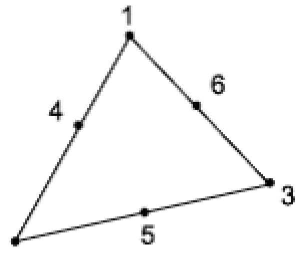


Figure 4 *Node numbering convention for DKMT element*

where,

$$f_i = \xi_i,\ P_4 = 4\xi_1\xi_2,\ P_5 = 4\xi_2\xi_3,\ P_6 = 4\xi_3\xi_1\ \left(\xi_1 + \xi_2 + \xi_3 = 1\right) \qquad (i = 1, 2, 3)$$

Then, the curvature vector is calculated by,

$$\underline{\kappa} = \mathbf{B}_{b\beta}\mathbf{u} + \mathbf{B}_{b\Delta\beta}\Delta\underline{\beta}$$

where:

$$\mathbf{B}_{b\beta} = \frac{1}{2A}\begin{bmatrix} 0 & -y_{32} & 0 & 0 & -y_{13} & 0 & 0 & -y_{21} & 0 \\ 0 & 0 & x_{32} & 0 & 0 & x_{13} & 0 & 0 & x_{21} \\ 0 & x_{32} & -y_{32} & 0 & x_{13} & -y_{13} & 0 & x_{21} & -y_{21} \end{bmatrix}$$

$$\mathbf{B}_{b\Delta\beta} = \frac{1}{2A}\begin{bmatrix} & -\left(\frac{\partial P_k}{\partial \xi_2} y_{13} + \frac{\partial P_k}{\partial \xi_3} y_{21}\right)c_k & \\ \cdots & \left(\frac{\partial P_k}{\partial \xi_2} x_{13} + \frac{\partial P_k}{\partial \xi_3} x_{21}\right)s_k & \cdots \\ & \left(\frac{\partial P_k}{\partial \xi_2} x_{13} + \frac{\partial P_k}{\partial \xi_3} x_{21}\right)c_k - \left(\frac{\partial P_k}{\partial \xi_2} y_{13} + \frac{\partial P_k}{\partial \xi_3} y_{21}\right)s_k & \end{bmatrix} \left(k = 4, 5, 6\right)$$

$$\Delta\underline{\beta} = \left[\Delta\beta_{s4} \quad \Delta\beta_{s5} \quad \Delta\beta_{s6}\right]^T$$

Also, the shear strains are calculated by,

$$\underline{\gamma} = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \mathbf{B}_{s\Delta\beta}\Delta\underline{\beta}$$

where

$$\mathbf{B}_{s\Delta\beta} = \frac{2}{3}\begin{bmatrix} \left(\frac{s_5}{A_2}\xi_2 - \frac{s_6}{A_1}\xi_1\right)\phi_4 & \left(\frac{s_6}{A_3}\xi_3 - \frac{s_4}{A_2}\xi_2\right)\phi_5 & \left(\frac{s_4}{A_1}\xi_1 - \frac{s_5}{A_3}\xi_3\right)\phi_6 \\ \left(\frac{c_6}{A_1}\xi_1 - \frac{c_5}{A_2}\xi_2\right)\phi_4 & \left(\frac{c_4}{A_2}\xi_2 - \frac{c_6}{A_3}\xi_3\right)\phi_5 & \left(\frac{c_5}{A_3}\xi_3 - \frac{c_4}{A_1}\xi_1\right)\phi_6 \end{bmatrix}$$

in which

$$\phi_k = \frac{2}{k(1-\nu)}\left(\frac{h^2}{l_{ij}^2}\right),\ A_1 = c_4 s_6 - c_6 s_4,\ A_2 = c_5 s_4 - c_4 s_5,\ A_3 = c_6 s_5 - c_5 s_6$$

$$A_1 = c_4 s_6 - c_6 s_4,\ A_2 = c_5 s_4 - c_4 s_5,\ A_3 = c_6 s_5 - c_5 s_6$$

Accordingly, the strain-displacement matrix $\mathbf{B}$ is given by,

$$\underline{\kappa} = \mathbf{B}_b\mathbf{u} = \left(\mathbf{B}_{b\beta} + \mathbf{B}_{b\Delta\beta}\mathbf{A}_n\right)\mathbf{u}$$
$$\underline{\gamma} = \mathbf{B}_s\mathbf{u} = \mathbf{B}_{s\Delta\beta}\mathbf{A}_n\mathbf{u}$$

where, $\mathbf{A}_n$ is the transformation matrix, which relates $\Delta\underline{\beta} = \mathbf{A}_n\mathbf{u}$ and is given by,

$$\mathbf{A}_n = \mathbf{A}_{\Delta\beta}^{-1}\mathbf{A}_w$$

$$\mathbf{A}_{\Delta\beta} = \begin{bmatrix} \frac{2}{3}l_{12}(1+\phi_4) & 0 & 0 \\ 0 & \frac{2}{3}l_{23}(1+\phi_5) & 0 \\ 0 & 0 & \frac{2}{3}l_{31}(1+\phi_6) \end{bmatrix}$$

$$\mathbf{A}_w = \begin{bmatrix} 1 & -\frac{x_{21}}{2} & -\frac{y_{21}}{2} & -1 & -\frac{x_{21}}{2} & -\frac{y_{21}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{x_{32}}{2} & -\frac{y_{32}}{2} & -1 & -\frac{x_{32}}{2} & -\frac{y_{32}}{2} \\ -1 & -\frac{x_{13}}{2} & -\frac{y_{13}}{2} & 0 & 0 & 0 & 1 & -\frac{x_{13}}{2} & -\frac{y_{13}}{2} \end{bmatrix}$$

Therefore, the final stiffness matrix is given by,

$$\mathbf{K} = \mathbf{K}_b + \mathbf{K}_s$$
$$\mathbf{K}_b = \int_A \mathbf{B}_b^T\mathbf{D}_b\mathbf{B}_b dA,\quad \mathbf{K}_s = \int_A \mathbf{B}_s^T\mathbf{D}_s\mathbf{B}_s dA$$

Also, the bending moments and shear forces at any point in the element are defined as,

$$\mathbf{M} = \mathbf{D}_b\mathbf{B}_b\mathbf{u}$$
$$\mathbf{T} = \mathbf{D}_s\mathbf{B}_s\mathbf{u}$$

The formulation procedure of the **DKMQ** element is quite similar to that used for the **DKMT** element. Except, some differences exist due to the differences in the number of nodes and shape functions. The matrices for the **DKMQ** element corresponding to those for the **DKMT** element are as follows:

$$\begin{cases} f_1 = \frac{1}{4}(1-\xi)(1-\eta) & P_5 = \frac{1}{2}(1-\xi^2)(1-\eta) \\ f_2 = \frac{1}{4}(1+\xi)(1-\eta) & P_6 = \frac{1}{2}(1+\xi)(1-\eta^2) \\ f_3 = \frac{1}{4}(1+\xi)(1+\eta) & P_7 = \frac{1}{2}(1-\xi^2)(1+\eta) \\ f_4 = \frac{1}{4}(1-\xi)(1+\eta) & P_8 = \frac{1}{2}(1-\xi)(1-\eta^2) \end{cases}$$

$$\mathbf{B}_{b\beta} = \begin{bmatrix} & 0 & a_i & 0 & \\ \cdots & 0 & 0 & b_i & \cdots \\ & 0 & b_i & a_i & \end{bmatrix}, \quad \begin{cases} a_i = j_{11}\dfrac{\partial f_i}{\partial \xi} + j_{12}\dfrac{\partial f_i}{\partial \eta} \\ b_i = j_{21}\dfrac{\partial f_i}{\partial \xi} + j_{22}\dfrac{\partial f_i}{\partial \eta} \end{cases}, \; (i = 1,2,3,4)$$

$$\mathbf{B}_{b\Delta\beta} = \begin{bmatrix} & \left(\dfrac{\partial P_k}{\partial \xi} j_{11} + \dfrac{\partial P_k}{\partial \eta} j_{12}\right) c_k & \\ \cdots & \left(\dfrac{\partial P_k}{\partial \xi} j_{21} + \dfrac{\partial P_k}{\partial \eta} j_{22}\right) s_k & \cdots \\ & \left(\dfrac{\partial P_k}{\partial \xi} j_{21} + \dfrac{\partial P_k}{\partial \eta} j_{22}\right) c_k - \left(\dfrac{\partial P_k}{\partial \xi} j_{11} + \dfrac{\partial P_k}{\partial \eta} j_{12}\right) s_k & \end{bmatrix} (k = 5,6,7,8)$$

$$\Delta\underline{\beta} = \left[\Delta\beta_{s5} \quad \Delta\beta_{s6} \quad \Delta\beta_{s7} \quad \Delta\beta_{s8}\right]^T$$

$$\mathbf{B}_{s\Delta\beta} = \frac{1}{6}\begin{bmatrix} -j_{11}(1-\eta)l_{12}\phi_5 & -j_{12}(1+\xi)l_{23}\phi_6 & j_{11}(1+\eta)l_{34}\phi_7 & j_{12}(1-\xi)l_{41}\phi_8 \\ -j_{21}(1-\eta)l_{12}\phi_5 & -j_{22}(1+\xi)l_{23}\phi_6 & j_{21}(1+\eta)l_{34}\phi_7 & j_{22}(1-\xi)l_{41}\phi_8 \end{bmatrix}$$

$$\phi_k = \frac{2}{k(1-\nu)}\left(\frac{h^2}{l_{ij}^2}\right) \qquad (k = 5,6,7,8)$$

where, j_{11}, j_{12}, j_{21} and j_{22} are the components of the inverse Jacobian matrix.

The transformation matrix $\mathbf{A}_n$ takes the following form:

$$\mathbf{A}_n = \mathbf{A}_{\Delta\beta}^{-1}\mathbf{A}_w$$

$$\mathbf{A}_{\Delta\beta} = \begin{bmatrix} \frac{2}{3}l_{12}\left(1+\phi_5\right) & 0 & 0 & 0 \\ 0 & \frac{2}{3}l_{23}\left(1+\phi_6\right) & 0 & 0 \\ 0 & 0 & \frac{2}{3}l_{34}\left(1+\phi_7\right) & 0 \\ 0 & 0 & 0 & \frac{2}{3}l_{41}\left(1+\phi_8\right) \end{bmatrix}$$

$$\mathbf{A}_w = \begin{bmatrix} 1 & -\frac{x_{21}}{2} & -\frac{y_{21}}{2} & -1 & -\frac{x_{21}}{2} & -\frac{y_{21}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{x_{32}}{2} & -\frac{y_{32}}{2} & -1 & -\frac{x_{32}}{2} & -\frac{y_{32}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{x_{43}}{2} & -\frac{y_{43}}{2} & -1 & -\frac{x_{43}}{2} & -\frac{y_{43}}{2} \\ -1 & -\frac{x_{14}}{2} & -\frac{y_{14}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{x_{14}}{2} & -\frac{y_{14}}{2} \end{bmatrix}$$