

Material Nonlinear Analysis

A fundamental difference between elastic and plastic material behaviors is that no permanent deformations occur in the structure in elastic behavior, whereas permanent or irreversible deformations occur in the structure in plastic behavior.

■ Plasticity theory

The components of static plastic strain are constituted by the following assumptions:

- Constitutive response is independent of the rate of deformation.
- Elastic response is not influenced by plastic deformation.
- Additive strain decomposition into elastic and plastic parts is defined by

$$\underline{\varepsilon} = \underline{\varepsilon}^e + \underline{\varepsilon}^p \quad (1)$$

where,

- $\underline{\varepsilon}$: total strains
- $\underline{\varepsilon}^e$: elastic strains
- $\underline{\varepsilon}^p$: plastic strains

And the following basic concepts are used to formulate the equations:

- Yield criteria to define the initiation of plastic deformation
- Flow rule to define the plastic straining
- Hardening rule to define the evolution of the yield surface with plastic straining

Yield criteria

The yield function (or loading function), F , which defines the limit for the range of elastic response, is as follows (Fig. 2.16):

$$F(\underline{\sigma}, \underline{\varepsilon}^p, \kappa) = \sigma_e(\underline{\sigma}, \underline{\varepsilon}^p) - \kappa(\varepsilon_p) \leq 0 \quad (2)$$

where,

$\underline{\sigma}$: current stresses

σ_e : equivalent or effective stress

κ : hardening parameter which is a function of ε^p

ε_p : equivalent plastic strain

In classical plasticity theory, a state of stress at which the value of the yield function becomes positive is not admissible. When yielding occurs, the state of stress is corrected by scaling plastic strains until the yield function is reduced to zero. This process is known as the plastic corrector phase or return mapping.

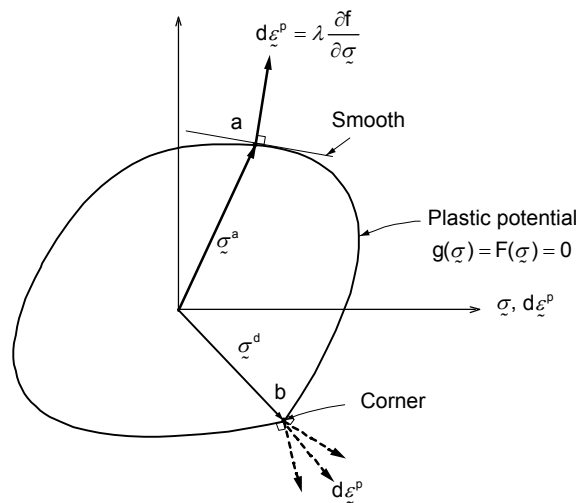


Fig 2.16 Geometric illustration of associated flow rule and singularity

Flow rule

The flow rule defines the plastic straining, which is expressed as follows (Fig 2.16):

$$d\tilde{\varepsilon}^p = d\lambda \frac{\partial g}{\partial \tilde{\sigma}} = d\lambda \mathbf{b} \quad (3)$$

where,

$\frac{\partial g}{\partial \tilde{\sigma}}$: the direction of plastic straining

$d\lambda$: plastic modulus which identifies the magnitude of plastic straining

The function g is termed as the 'plastic potential' function, which is generally defined in terms of stress invariants. If $g=F$, it is termed as 'associated flow rule', and if $g \neq F$, it is referred to as 'non-associative flow rule'.

The associated flow rule is adopted for all the yield criteria of MIDAS programs. As the direction of the plastic strain vector is normal to the yield surface, the above equation can be expressed as follows:

$$d\tilde{\varepsilon}^p = d\lambda \frac{\partial F}{\partial \tilde{\sigma}} = d\lambda \mathbf{a} \quad (4)$$

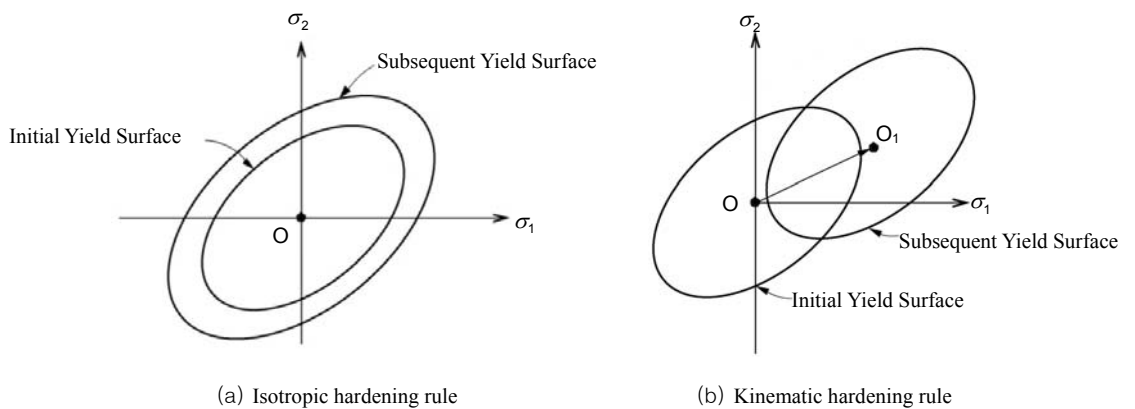
The corner or the flat surface in Fig 2.16 represents a singular point, which can not uniquely determine the direction of plastic flow. These points require special consideration.

Hardening rule

The hardening rule defines the expansion and translation of the yield surface with plastic straining as the material yields.

Depending on the method of defining the effective plastic strain, the hardening rule is classified into 'strain hardening' and 'work hardening'. The strain hardening is defined by the hypothesis of plastic incompressibility, and as such it is appropriate for a material model, which is not influenced by hydrostatic stress. Accordingly, work hardening, which is defined by plastic work, is more generally applicable than strain hardening.

Also, depending on the type of change of yield surface, the hardening rule is classified into 'isotropic hardening', 'kinematic hardening' and 'mixed hardening' (Fig. 2.17).



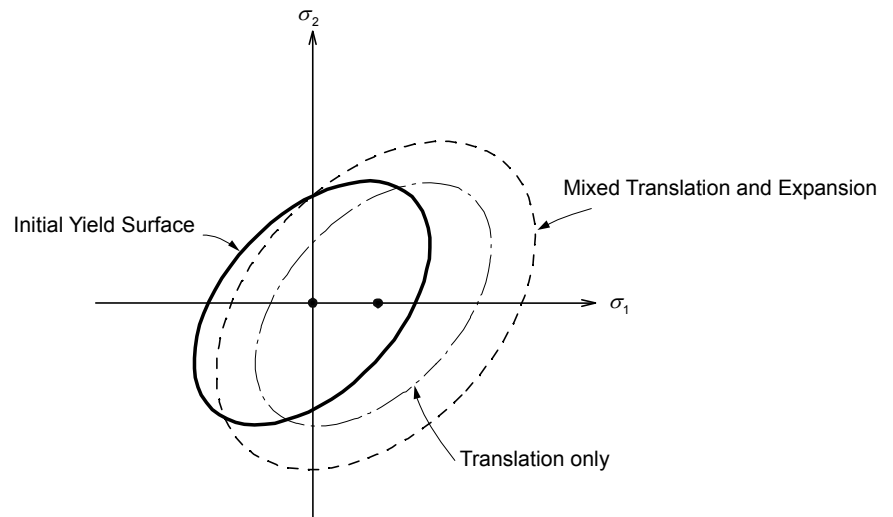


Fig 2.17 Mixed hardening with kinematic hardening

■ Classification by the method of defining the effective plastic strain

1. Strain hardening

The effective plastic strain in strain hardening is defined as follows:

$$d\varepsilon_p = \sqrt{\frac{2}{3}(d\varepsilon^p)^T d\varepsilon^p} = \sqrt{\frac{2}{3}\mathbf{a}^T \mathbf{a} d\lambda} \quad (5)$$

The effective plastic strain is derived from transforming the norm of plastic strains to conform to uniaxial strain with the assumption that there is no volumetric plastic deformation. Although this is applicable in principle only to Tresca or von Mises, it is often applied to other cases because of numerical convenience.

2. Work hardening

The increment of plastic work is as follows:

$$dW_p = \underline{\sigma}^T d\underline{\varepsilon}^p = d\lambda \underline{\mathbf{a}}^T \underline{\sigma} \quad (6)$$

In the case of uniaxial strain, the increment of the plastic work is expressed as,

$$dW_p = \sigma_1 d\varepsilon_1 = \sigma_e d\varepsilon_p \quad (7)$$

Hence the effective plastic strain pertaining to work hardening is defined as follows:

$$d\varepsilon_p = \frac{\underline{\mathbf{a}}^T \underline{\sigma}}{\sigma_e} d\lambda \quad (8)$$

■ Classification by the types of change of yield surface

1. Perfectly plastic

A perfectly plastic material does not change the yield surface even after plastic deformation has taken place. The yield function then can be expressed as follows:

$$F(\underline{\sigma}, \kappa) = \sigma_e(\underline{\sigma}) - \kappa \quad (9)$$

where,

κ : constant

2. Isotropic hardening

In the case of isotropic hardening, the yield surface expands uniformly as shown in Fig. 2.18(a). The yield function can be expressed as follows:

$$F(\underline{\sigma}, \kappa) = \sigma_e(\underline{\sigma}) - \kappa(\varepsilon_p) \quad (10)$$

3. Kinematic hardening

In the case of kinematic hardening, the size of the yield surface remains unchanged and the center location of the yield surface is shifted as shown in Fig.

2.18(b). The yield function can be expressed as follows:

$$F(\underline{\sigma}, \underline{\alpha}, \kappa) = \sigma_e(\underline{\sigma} - \underline{\alpha}) - \kappa \quad (11)$$

where,

$\underline{\alpha}$: the center coordinates of yield surface

κ : constant

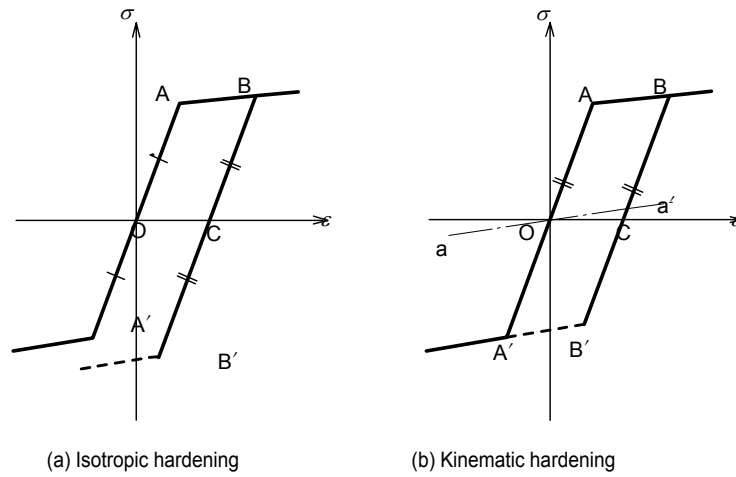


Fig. 2.18 Hardening rule in 1-dimension

In kinematic hardening, it becomes important to determine the center coordinates of the subsequent yield surface, $\underline{\alpha}$. In order to determine the “kinematic shift”, $\underline{\alpha}$, there exist Prager’s hardening rule, Ziegler’s hardening rule, etc.

The Prager’s hardening rule can be expressed as,

$$d\underline{\alpha} = C_p d\underline{\varepsilon}^p = C_p \underline{a} d\lambda \quad (12)$$

where,

C_p : Prager’s hardening coefficient

This method may present some problems when it is used in the sub space of

stress. For example, $d\alpha$ may not be 0 even any component of stresses is 0, which may not only present translation of the yield surface. The Ziegler's hardening rule on the other hand assumes that the rate of translation of the center, $d\alpha$, takes place in the direction of the reduced-stress vector, $\underline{\sigma} - \underline{\alpha}$. Hence, it presents no such problem. This hardening rule is expressed as follows:

$$d\alpha = d\mu(\underline{\sigma} - \underline{\alpha}) = C_z d\varepsilon_p (\underline{\sigma} - \underline{\alpha}) \quad (13)$$

where,

C_z : Ziegler's hardening coefficient

4. Mixed hardening

Mixed hardening is a hardening type, which represents the mix of isotropic hardening and kinematic hardening, which is expressed as follows:

$$F(\underline{\sigma}, \alpha, \kappa) = \sigma_e(\underline{\sigma} - \underline{\alpha}) - \kappa(\varepsilon_p) \quad (14)$$

■ Constitutive equations

Standard plastic constitutive equations are formulated as below. Stress increments are determined by the elastic part of the strain increments.

That is,

$$d\underline{\sigma} = \underline{\mathbf{D}}^e (d\underline{\varepsilon} - d\underline{\varepsilon}^p) = \underline{\mathbf{D}}^e (d\underline{\varepsilon} - d\lambda \underline{\mathbf{a}}) \quad (15)$$

where,

$\underline{\mathbf{D}}^e$: elastic constitutive matrix

In order to always maintain the stresses on the yield surface, the following consistency condition needs to be satisfied.

$$dF = \frac{\partial F}{\partial \underline{\sigma}} d\underline{\sigma} + \frac{\partial F}{\partial \underline{\varepsilon}^p} d\underline{\varepsilon}^p + \frac{\partial F}{\partial \kappa} d\kappa = \underline{\mathbf{a}}^T \underline{\mathbf{D}}^e d\underline{\varepsilon} - (\underline{\mathbf{a}}^T \underline{\mathbf{D}}^e \underline{\mathbf{a}} + h) d\lambda = 0 \quad (16)$$

where, h : plastic hardening modulus $(= \frac{d\sigma_e}{d\varepsilon_p})$

Accordingly, the rate of infinitesimal stress increments can be obtained as follows:

$$d\tilde{\sigma} = \tilde{\mathbf{D}}^e d\tilde{\varepsilon} - d\lambda \tilde{\mathbf{D}}^e \tilde{\mathbf{a}}$$

$$d\tilde{\sigma} = \left(\tilde{\mathbf{D}}^e - \frac{\tilde{\mathbf{D}}^e \tilde{\mathbf{a}} \tilde{\mathbf{a}}^T \tilde{\mathbf{D}}^{eT}}{\tilde{\mathbf{a}}^T \tilde{\mathbf{D}}^e \tilde{\mathbf{a}} + h} \right) d\tilde{\varepsilon} \quad (17)$$

When the full Newton-Raphson iteration procedure is used and if a consistent stiffness matrix is used, a much faster convergence can be achieved due to the second-order convergence characteristic of the Newton-Raphson iteration procedure.

$$d\tilde{\sigma} = \tilde{\mathbf{D}}^e d\tilde{\varepsilon} - d\lambda \tilde{\mathbf{D}}^e \tilde{\mathbf{a}} - \Delta \lambda \tilde{\mathbf{D}}^e \frac{\partial \tilde{\mathbf{a}}}{\partial \tilde{\sigma}} d\tilde{\sigma}$$

$$d\tilde{\sigma} = \left(\tilde{\mathbf{R}} - \frac{\tilde{\mathbf{R}} \tilde{\mathbf{a}} \tilde{\mathbf{a}}^T \tilde{\mathbf{R}}^T}{\tilde{\mathbf{a}}^T \tilde{\mathbf{R}} \tilde{\mathbf{a}} + h} \right) d\tilde{\varepsilon} \quad (18)$$

where,

$$\tilde{\mathbf{R}} = \left(\mathbf{I} + d\lambda \tilde{\mathbf{D}}^e \frac{\partial \tilde{\mathbf{a}}}{\partial \tilde{\sigma}} \right)^{-1} \tilde{\mathbf{D}}^e = \left(\mathbf{I} + d\lambda \tilde{\mathbf{D}}^e \tilde{\mathbf{A}} \right)^{-1} \tilde{\mathbf{D}}^e$$

■ Stress integration

The following two methods can be used for the integration of stresses:

- Explicit forward Euler algorithm with sub-incrementation (Fig. 2.19 & 20)
- Implicit backward Euler algorithm (Fig. 2.21)

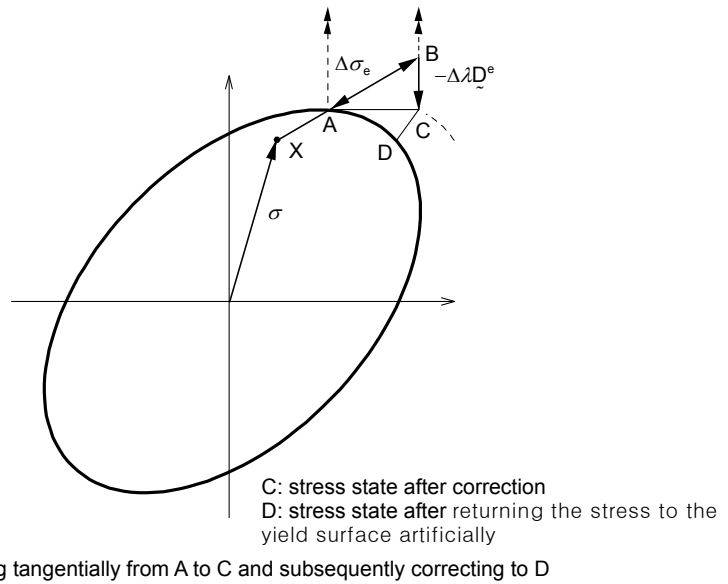
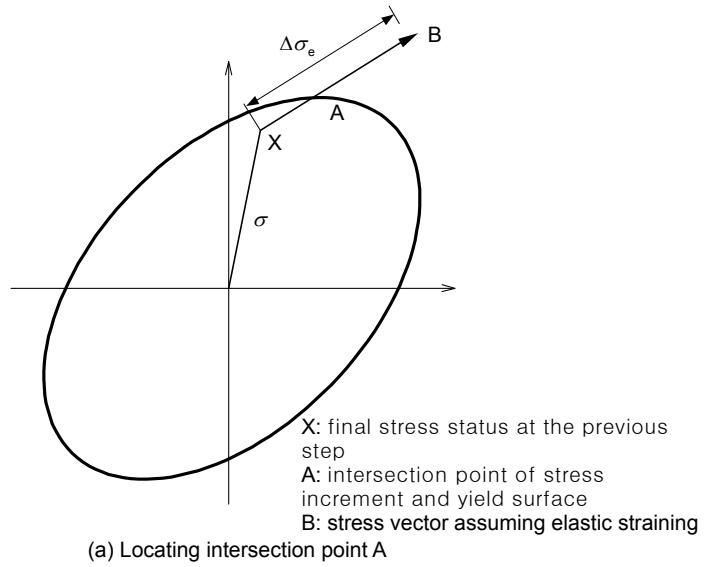


Fig. 2.19 Explicit forward-Euler procedure

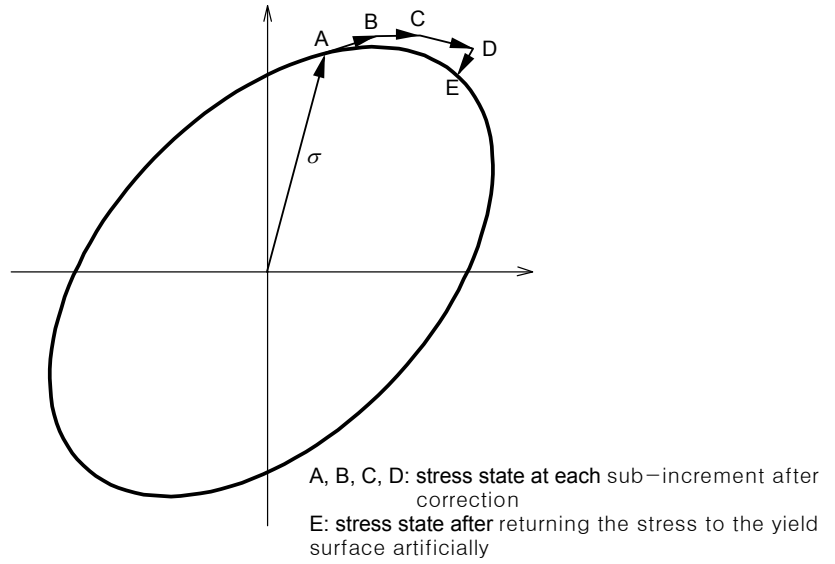


Fig. 2.20 Sub-incrementation in Explicit forward-Euler procedure

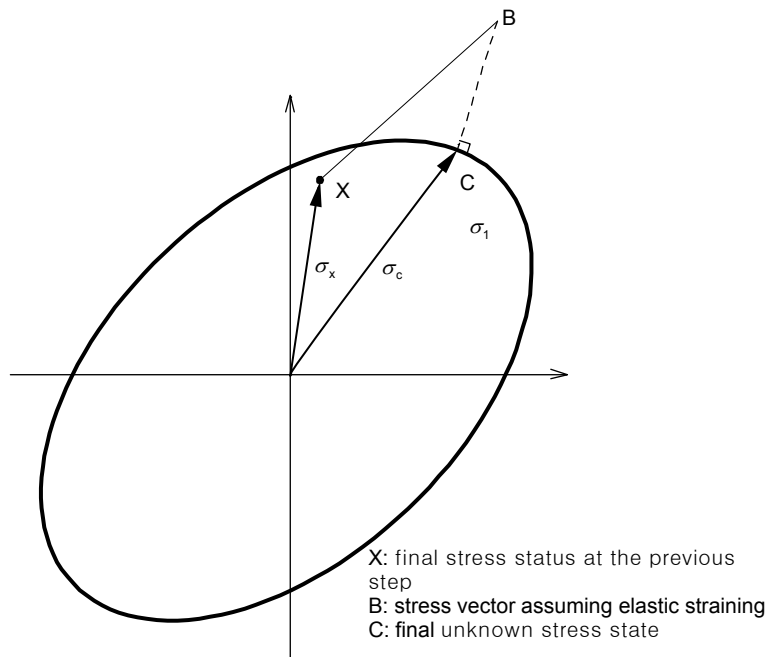


Fig. 2.21 Implicit backward-Euler procedure

In the Forward-Euler algorithm, the hardening data and the direction of plastic flow are calculated at the intersection point, where elastic stress increments cross the yield surface (at point A in Fig. 2.20). Whereas in the Backward-Euler algorithm, they are calculated at the final stress point (at point B in Fig. 2.21).

The Forward-Euler algorithm is relatively simple, and the stresses are directly integrated. That is, it need not iterate at the Gauss points, but presents the following drawbacks:

- It is conditionally stable.
- Sub-increments are required while correcting the stresses to obtain allowable accuracy.
- An artificial returning scheme is required to correct the stress state for drift from the yield surface.

Also, this method does not permit formulating a consistent stiffness matrix.

The Implicit Backward-Euler algorithm is unconditionally stable and accurate without sub-increments or artificial returning. However for general yield criteria, iterations are required at the Gauss points. Because a consistent stiffness matrix can be formulated using this method, even if iterations are performed at the Gauss points, it is more efficient if the Newton-Raphson iteration procedure is used.

Steps for applying the Explicit forward-Euler procedure

1. Calculate strain increments.

$$d\varepsilon = \mathbf{B}d\mathbf{u} \quad (19)$$

where,

\mathbf{B} : strain-displacement relation matrix

$d\varepsilon$: the changes of displacements

2. Calculate elastic stresses assuming elastic straining (at point B in Fig. 2.19(a)).

$$\begin{aligned} d\bar{\sigma} &= \mathbf{D}^e d\bar{\varepsilon} \\ \bar{\sigma}_B &= \bar{\sigma}_X + d\bar{\sigma} \end{aligned} \quad (20)$$

The Fig. 2.19 should be referenced for the subscripts in the equations above and below.

3. If the calculated stresses remain on the yield surface, stress correcting is completed. If the stresses exist beyond the yield surface, the stresses are returned to the yield surface by plastic straining.
4. Subsequently, the stresses at the intersection point are calculated. Elastic stress increments are divided into allowable stress increments and unallowable stress increments; whereas, stresses at the intersection point are calculated by the following expressions (point A in Fig. 2.19(a)):

$$\begin{aligned} F(\bar{\sigma}_X + (1-r)d\bar{\sigma}) &= 0 \\ r &= \frac{F_B}{F_B - F_X} \end{aligned} \quad (21)$$

5. Further straining would cause the stress location to traverse the yield surface. This is approximated by sub-dividing the unallowable stress increments, $rd\bar{\sigma}$, into the m number of small stress increments (Fig. 2.20). The number of sub-increments, m is directly related to the magnitude of the error resulted from a one step return, which is calculated as,

$$m = \text{INT} \left(8(\sigma_{eB} - \sigma_{eA}) / \sigma_{eA} \right) + 1 \quad (22)$$

6. If the final stress state does not lie on the yield surface, the following method of artificial returning is used to return the stress to the yield stress (point E in Fig. 2.20).

$$\begin{aligned} \delta\lambda_C &= \frac{F_C}{\mathbf{a}_C^T \mathbf{D}^e \mathbf{a}_C + h} \\ \bar{\sigma}_D &= \bar{\sigma}_C - \delta\lambda_C \mathbf{D}^e \mathbf{a}_C \end{aligned} \quad (23)$$

Notes

- The shape of the yield surface is corrected using the hardening rule at the end of each sub-increment.
- Unloading is assumed to be elastic.

Steps for applying the Implicit backward-Euler procedure

The final stress in the Backward-Euler algorithm is calculated by the following equation:

$$\underline{\sigma}_C = \underline{\sigma}_B - d\lambda \underline{\mathbf{D}}^e \underline{\mathbf{a}}_C \quad (24)$$

The Fig. 2.21 should be referenced for the subscripts.

Since the point C in the equation (24) is unknown, the Newton iteration is used to evaluate the unknowns. Accordingly, a vector, $\underline{\mathbf{r}}$, is set up to represent the difference between the current stresses and the backward-Euler stresses.

$$\underline{\mathbf{r}} = \underline{\sigma}_C - (\underline{\sigma}_B - d\lambda \underline{\mathbf{D}}^e \underline{\mathbf{a}}_C) \quad (25)$$

Now, iterations are introduced in order to reduce $\underline{\mathbf{r}}$ to 0 while the final stresses should satisfy the yield criterion, $f=0$. Using assumed elastic stresses, a truncated Taylor expansion is applied to the equation (25) to produce a new residual,

$$\underline{\mathbf{r}}_i = \underline{\mathbf{r}}_o + \dot{\sigma} + \dot{\lambda} \underline{\mathbf{D}}^e \underline{\mathbf{a}} \quad (26)$$

where,

$\dot{\sigma}$: the change in σ

$\dot{\lambda}$: the change in $d\lambda$

Setting the above equation to 0, and solving it for $\dot{\sigma}$, we obtain the following:

$$\dot{\sigma} = -\underline{\mathbf{r}}_o - \dot{\lambda} \underline{\mathbf{D}}^e \underline{\mathbf{a}} \quad (27)$$

Similarly, a truncated Taylor expansion is applied to the yield function, which

results in the following:

$$F_{Cn} = F_{Co} + \frac{\partial F^T}{\partial \sigma} \dot{\sigma} + \frac{\partial F}{\partial \varepsilon_p} \dot{\varepsilon}_p = F_{Co} + \mathbf{a}_c^T \dot{\sigma} - h \dot{\lambda} = 0 \quad (28)$$

where,

ε_p : effective plastic strain

Hence, $\dot{\lambda}$ is obtained, and the final stress values can be obtained as well.

$$\dot{\lambda} = \frac{F_o - \mathbf{a}^T \mathbf{r}_o}{\mathbf{a}^T \mathbf{D}^e \mathbf{a} + h} \quad (29)$$

■ Plastic material models

The following 4 types of general plastic models are used:

- Tresca & von Mises – suitable for ductile materials such as metals, which exhibit plastic incompressibility (Fig. 2.22).
- Mohr-Coulomb & Drucker-Prager – suitable for materials such as concrete, rock and soils, which exhibit volumetric plastic deformations (Fig. 2.23).

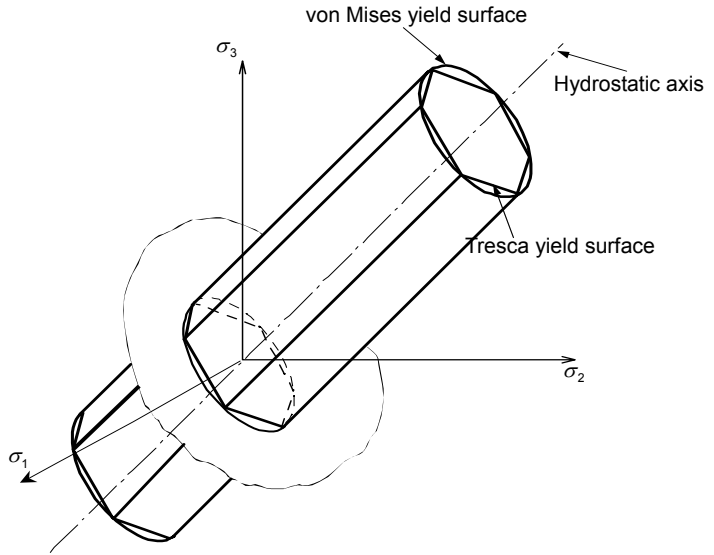


Fig 2.22 Tresca & von Mises yield criteria

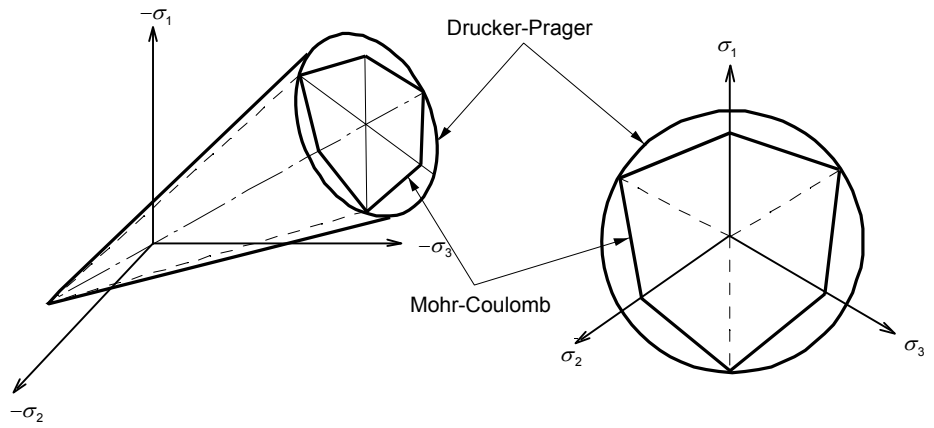


Fig 2.23 Mohr-Coulomb & Drucker-Prager yield criteria

Tresca criterion

The Tresca yield criterion is suitable for ductile materials such as metals, which exhibit little volumetric plastic deformations. The yielding of a material begins when the maximum shear stress reaches a specified value. So if the principal stresses are $\sigma_1, \sigma_2, \sigma_3$ ($\sigma_1 \geq \sigma_2 \geq \sigma_3$), the yield function becomes the equation (30).

$$F(\underline{\sigma}, \kappa) = \sigma_1 - \sigma_3 - \kappa(\varepsilon_p) \quad (30)$$

Numerical problems arise when the stress point lies at a singular point on the yield surface, which occurs when the lode angle θ approaches $\pm 30^\circ$. In such cases, the stress integration scheme must be corrected.

Von Mises criterion

The Von Mises criterion is a most widely used yield criterion for metallic materials. It is based on distortional strain energy, and the yield function is expressed as follows:

$$F(\underline{\sigma}, \kappa) = \sqrt{3J_2} - \kappa(\varepsilon_p) \quad (31)$$

where,

J_2 : second deviatoric stress invariant

Mohr-Coulomb criterion

The Mohr-Coulomb criterion is suitable for such materials as concrete, rock and soils, which exhibit volumetric plastic deformations. The Mohr-Coulomb yield criterion is a generalization of the Coulomb's friction rule, which is defined by,

$$F(\underline{\sigma}, \kappa) = \tau - (c - \sigma_n \tan \phi) \quad (32)$$

where,

τ : the magnitude of shearing stress

σ_n : normal stress

c : cohesion

ϕ : internal friction angle

The cohesion, c , and the internal friction angle, ϕ , are dependent upon the strain hardening parameter, κ .

Similar to the Tresca criterion, numerical problems occur when the stress point lies at a singular point on the yield surface. For the Mohr-Coulomb criterion, such numerical problems occur as the lode angle, θ , approaches $\pm 30^\circ$ or at the apex points. Hence, the stress integration scheme must be corrected for the two cases.

Drucker-Prager criterion

The Drucker-Prager criterion is suitable for such materials as soils, concrete and rock, which exhibit volumetric plastic deformations. This criterion is a smooth approximation of the Mohr-Coulomb criterion and is an expansion of the von Mises criterion. The yield function includes the effect of hydrostatic stress, which is defined as follows:

$$F(\boldsymbol{\sigma}, \kappa) = \frac{2 \sin \phi}{\sqrt{3}(3 - \sin \phi)} I_1 + \sqrt{J_2} - \frac{6c \cos \phi}{\sqrt{3}(3 - \sin \phi)} \quad (33)$$

where,

I : first stress invariant

For the Drucker-Prager criterion, Numerical problems occur when the stress point lies at the apex points of the yield surface.