Truss and Cable Elements

The Finite Element Library of the MIDAS Family Programs includes the following types of Truss and Cable Elements: general truss element, a tension-only truss element (with Hook ability), a compression-only truss element (with Gap ability), and a cable element (equivalent truss element & elastic element of a cable with sag "catenary cable").

A typical, three dimensional, truss element is shown Fig. 1. The formulation of truss element is based on the following assumption:

1. The element is a straight bar of uniform cross-section.
2. The element is capable of resisting only axial forces.
3. At the end of element can be member end-offsets along its centroidal axis.

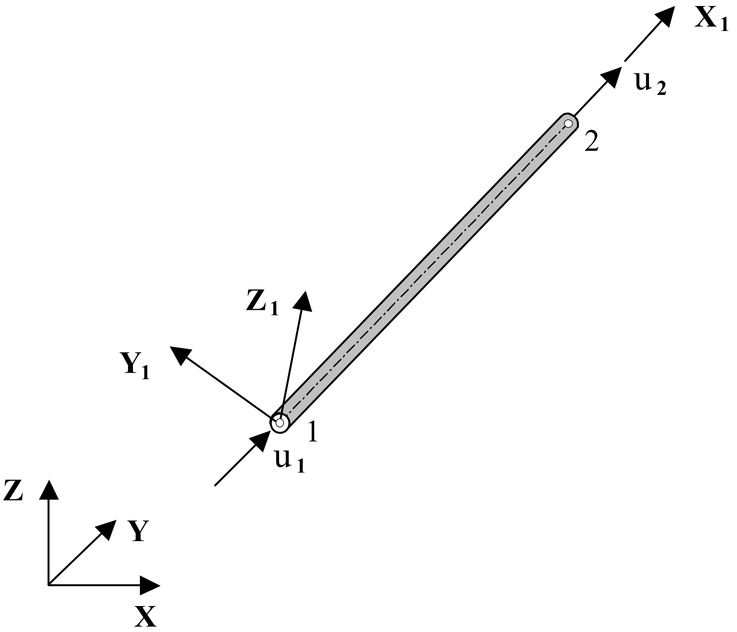


Figure 1 *Typical 3D Truss Element*

The formulation procedure for a general truss element will be illustrated on the basis of 2D truss element. The nodal degrees of freedom (DOF's) in local axes and the positive sign convention are shown in Fig. 2.

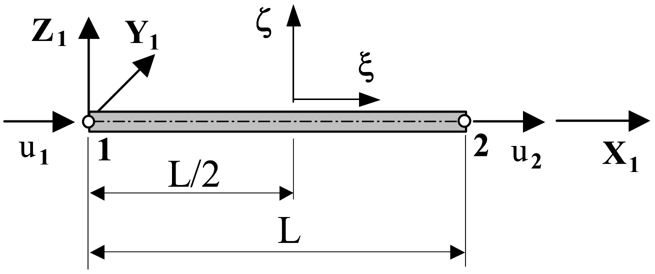


Figure 2 *A 2D Truss Element*

The element displacement field is defined in terms of nodal DOF's by the following function:

$$u = \frac{1}{2}(1-\xi)u_1 + \frac{1}{2}(1+\xi)u_2 \tag{1}$$

where

u_i = displacement along local element X_1 − axis at i − th node

ξ = natural coordinate (see Fig.1) of the element ξ ($-1 \le \xi \le +1$).

Axial displacement defined by Eq. (1) can be rewritten in terms of local **x** coordinate as:

$$u = \frac{1-x}{L}u_1 + \frac{x}{L}u_2 \tag{2}$$

In matrix form Eq. (2) can be expressed as

$$u = \begin{bmatrix} 1-\dfrac{x}{L} & \dfrac{x}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \mathbf{fq} \tag{3}$$

where

$\mathbf{f}$ = matrix of shape functions

$\mathbf{q}$ = vector of nodal displacements

The strain-displacement relationships for the truss element consist of one derivative

$$\boldsymbol{\varepsilon} = \varepsilon_x = \mathbf{du} = \frac{du}{dx} = \frac{d\mathbf{f}}{dx}\mathbf{q} = \mathbf{Bq} \tag{4}$$

where

$\boldsymbol{\varepsilon}$ = vector of strains

$\mathbf{d}$ = vector of derivative operators

$\mathbf{B}$ = the strain-displacement matrix

Hence, the strain-displacement matrix is presented as,

$$\mathbf{B} = \frac{d\mathbf{f}}{dx} = \frac{1}{L}\begin{bmatrix}-1 & 1\end{bmatrix} \tag{5}$$

Similarly, stress-strain relationships become merely,

$$\boldsymbol{\sigma} = \sigma_x = \mathbf{D}\boldsymbol{\varepsilon} = E\varepsilon_x = \mathbf{DBq} \tag{6}$$

in which

$\boldsymbol{\sigma}$ = vector of stresses

$\mathbf{D}$ = elasticity (constitutive) matrix.

Thus, the elasticity matrix is

$$\mathbf{D} = E \tag{7}$$

where E is the Young modulus

Then, the element stiffness matrix of a prismatic truss element can be evaluated as follows:

$$\mathbf{K} = \int_V \mathbf{B}^T\mathbf{DB}dV = \frac{E}{L^2}\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix}\int_0^L\int_A dAdx = \frac{EA}{L}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} \tag{8}$$

The stiffness matrices for a tension-only truss element and a compression-only truss element are the same as above. But, they can or cannot exhibit the stiffness depending on the condition defined by the sign of the member forces or relative displacements.

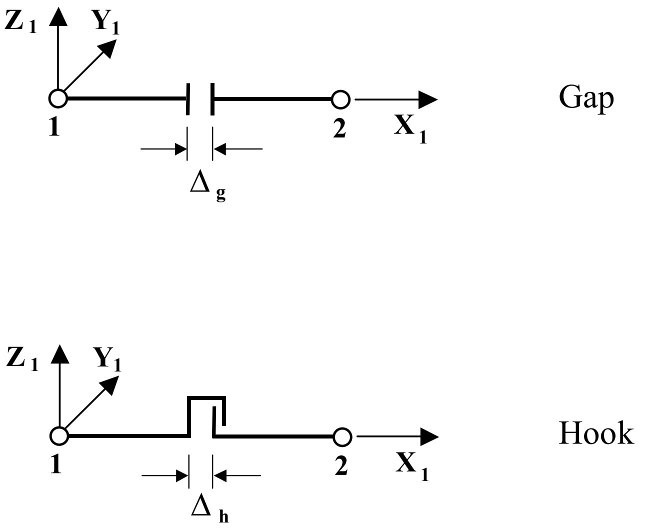


Figure 3 *Truss Element with Hook and Gap Propetries*

Thus, the tension-only truss element is capable of resisting only an axial tension force. Therefore, if a member force is positive (tension), it exhibits the stiffness previously described, and if a member force is negative (compression), its stiffness becomes zero. ‘Hook ability’ means that the element has a constant initial distance Δ_{h} , named as *hook distance* (see Fig.3); so only when the relative displacement is larger than or equal to Δ_{h} , the element exhibits its stiffness. That is,

$$\left(q_2 - q_1\right) \geq \Delta_{\mathrm{h}} \tag{9}$$

Conversely, a compression-only truss element transfers only an axial compression force. Therefore, only when a member force is negative, it exhibits the stiffness of Eq. (8). Similarly, it has ‘Gap ability’, so only when the relative displacement is smaller than or equal to a constant initial distance Δ_{g} , termed as *gap distance* (see Fig. 3), it exhibits its stiffness. That is,

$$\left(q_2 - q_1\right) \leq \Delta_{\mathrm{g}} \tag{10}$$

A typical, cable-equivalent truss element is shown Fig. 1. Te cable element, however, is capable of resisting only the axial tension force and is used to model structural behavior of the cable. Basic property of the cable is known as “stress stiffening” due the tension force applied to the element. It should be noted, that this element is formulated in such way, that when it is used in linear analysis it is automatically transformed in an equivalent truss element, and in geometric nonlinear analysis it acts as an elastic catenary cable element.

The stiffness of the equivalent truss element consists of a general elastic stiffness and the stiffness due to sag. The cable element has the same nodal DOF’s and forces as a general truss element and general elastic stiffness is the same as in Eq. (8)

$$K_{elastic} = \frac{EA}{L}$$

while the stiffness due to sag is given by,

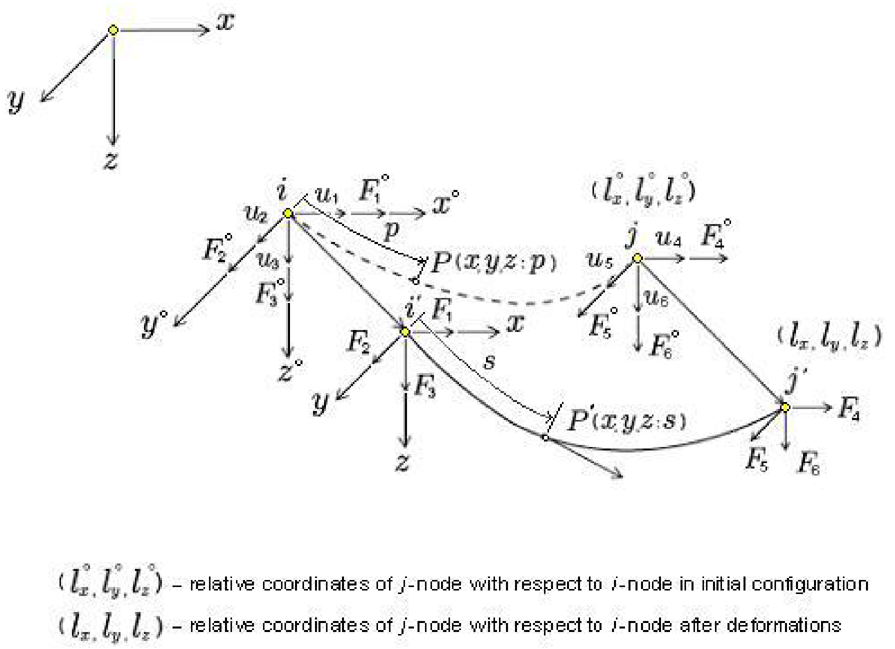
$$K_{sag} = \frac{12T^3}{w^2 L^3} \tag{11}$$

where, T is the tension force in the cable element, and w is the self-weight per unit length of the cable.
If we combine the elastic stiffness and the stiffness due to sag in series, the stiffness of the combined cable-equivalent truss becomes,

$$K_{comb} = \frac{1}{1/K_{elastic} + 1/K_{sag}} = \frac{EA}{L\left(1 + \dfrac{w^2 L^2 EA}{12T^3}\right)} \tag{12}$$

The elastic catenary cable element is based on Lagrangian formulation and accounts for geometric nonlinear effects caused by large displacement. Basic procedure in the formulation of this element is evaluation of tangent stiffness matrix. For the implemented cable element the tangent stiffness matrix is obtained as follows. Let’s consider a 2-node cable, located in the fixed coordinate system $x,\ y,\ z$, as shown in Fig. 4. Assume that in initial equilibrium configuration cable is subjected to six nodal forces $F_1^0, F_2^0, F_3^0, F_4^0, F_5^0, F_6^0$ and his initial properties are defined as follows: section area A_0, length L_0, and self-weight per unit length (weight density) w_0.

Assume that cable is subjected to a displacement field defined by the six displacement components: u_1, u_2 and u_3 at the node i and u_4, u_5 and u_6 at the node j, accompanying the nodal forces F_1, F_2, F_3, F_4, F_5 and F_6. The Lagrangian (curvilinear) coordinate of an arbitrary point P (x, y, z) on a cable element is s for the unstretched length, and p for the stretched length (see Fig. 4).



$(l_x^\circ, l_y^\circ, l_z^\circ)$ – relative coordinates of j-node with respect to i-node in initial configuration

(l_x, l_y, l_z) – relative coordinates of j-node with respect to i-node after deformations

Figure 4 *A Catenary (Cable) Element subjected to nodal displacements*

In this configuration of the cable element, the geometric constraints and the equilibrium conditions for tension forces are,

$$\left(\frac{dx}{dp}\right)^2+\left(\frac{dy}{dp}\right)^2+\left(\frac{dz}{dp}\right)^2=1 \tag{13}$$

$$\begin{aligned} T\left(\frac{dx}{dp}\right)&=-F_1\\ T\left(\frac{dy}{dp}\right)&=-F_2\\ T\left(\frac{dz}{dp}\right)&=-F_3-ws \end{aligned} \tag{14}$$

Accordingly, the tension force T at point P is given by,

$$T(s)=\left\{F_1^2+F_2^2+\left(F_3+ws\right)^2\right\}^{1/2} \tag{15}$$

The relationships between the undeformed Lagrangian coordinate s and Cartesian coordinate are,

$$\begin{aligned} x(s)&=\int\frac{dx}{ds}ds\\ y(s)&=\int\frac{dy}{ds}ds\\ z(s)&=\int\frac{dz}{ds}ds \end{aligned} \tag{16}$$

where,

$$\frac{dx}{ds}=\frac{dx}{dp}\frac{dp}{ds}=-\frac{F_1}{T}\frac{dp}{ds}=-\frac{F_1}{T}\left(\frac{T}{EA_0}+1\right)=-\left\{\frac{F_1}{EA_0}+\frac{F_1}{\left\{F_1^2+F_2^2+\left(F_3+ws\right)^2\right\}^{1/2}}\right\} \tag{17}$$

and the boundary conditions are,

$$\begin{aligned} x=0, y=0, z=0, p=0 \quad at \quad s=0\\ x=l_x, y=l_y, z=l_z, p=L \quad at \quad s=L_0 \end{aligned} \tag{18}$$

Therefore,

$$\begin{aligned} x(s)&=-\frac{F_1}{EA_0}s-\frac{F_1}{w}\left[\ln\left\{F_3+ws+\left(F_1^2+F_2^2+\left(F_3+ws\right)^2\right)^{1/2}\right\}-\ln\left\{F_3+\left(F_1^2+F_2^2+F_3^2\right)^{1/2}\right\}\right]\\ y(s)&=-\frac{F_2}{EA_0}s-\frac{F_2}{w}\left[\ln\left\{F_3+ws+\left(F_1^2+F_2^2+\left(F_3+ws\right)^2\right)^{1/2}\right\}-\ln\left\{F_3+\left(F_1^2+F_2^2+F_3^2\right)^{1/2}\right\}\right]\\ z(s)&=-\frac{F_3}{EA_0}s-\frac{F_1}{2EA_0}-\frac{1}{w}\left[\left\{F_1^2+F_2^2+\left(F_3+ws\right)^2\right\}^{1/2}-\left\{F_1^2+F_2^2+F_3^2\right\}^{1/2}\right] \end{aligned} \tag{19}$$

and,

$$l_x = -\frac{F_1 L_0}{EA_0} - \frac{F_1}{w}\left[\ln\left\{F_3 + wL_0 + \left(F_1^2 + F_2^2 + \left(F_3 + wL_0\right)^2\right)^{1/2}\right\} - \ln\left\{F_3 + \left(F_1^2 + F_2^2 + F_3^2\right)^{1/2}\right\}\right]$$

$$l_y = -\frac{F_2 L_0}{EA_0} - \frac{F_2}{w}\left[\ln\left\{F_3 + wL_0 + \left(F_1^2 + F_2^2 + \left(F_3 + wL_0\right)^2\right)^{1/2}\right\} - \ln\left\{F_3 + \left(F_1^2 + F_2^2 + F_3^2\right)^{1/2}\right\}\right] \quad (20)$$

$$l_z = -\frac{F_3 L_0}{EA_0} - \frac{wL_0^2}{2EA_0} - \frac{1}{w}\left[\left\{F_1^2 + F_2^2 + \left(F_3 + wL_0\right)^2\right\}^{1/2} - \left\{F_1^2 + F_2^2 + F_3^2\right\}^{1/2}\right]$$

The equilibrium conditions of the nodal forces and the compatibility conditions of nodal displacements are,

$$\begin{aligned} F_4 &= -F_1 \\ F_5 &= -F_2 \\ F_6 &= -F_3 - w_0 L_0 \\ l_x &= l_x^0 - u_1 + u_4 = f\left(F_1, F_2, F_3\right) \\ l_y &= l_y^0 - u_2 + u_5 = g\left(F_1, F_2, F_3\right) \\ l_z &= l_z^0 - u_3 + u_6 = h\left(F_1, F_2, F_3\right) \end{aligned} \quad (21)$$

Differentiating both sides of Eq. (21), the relationships between nodal forces and changes of cable lengths can be presented as:

$$\begin{aligned} dl_x &= \frac{\partial f}{\partial F_1} dF_1 + \frac{\partial f}{\partial F_2} dF_2 + \frac{\partial f}{\partial F_3} dF_3 \\ dl_y &= \frac{\partial g}{\partial F_1} dF_1 + \frac{\partial g}{\partial F_2} dF_2 + \frac{\partial g}{\partial F_3} dF_3 \\ dl_z &= \frac{\partial h}{\partial F_1} dF_1 + \frac{\partial h}{\partial F_2} dF_2 + \frac{\partial h}{\partial F_3} dF_3 \end{aligned} \quad (22)$$

or in matrix form

$$\begin{Bmatrix} dl_x \\ dl_y \\ dl_z \end{Bmatrix} = \mathbf{F} \begin{Bmatrix} dF_1 \\ dF_2 \\ dF_3 \end{Bmatrix} \quad (22a)$$

where $\mathbf{F}$ is, so called, nodal flexibility matrix defined as follows:

$$\mathbf{F} = \begin{bmatrix} \frac{\partial f}{\partial F_1} & \frac{\partial f}{\partial F_2} & \frac{\partial f}{\partial F_3} \\ \frac{\partial g}{\partial F_1} & \frac{\partial g}{\partial F_2} & \frac{\partial g}{\partial F_3} \\ \frac{\partial h}{\partial F_1} & \frac{\partial h}{\partial F_2} & \frac{\partial h}{\partial F_3} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \quad (22b)$$

Rewriting Eq. (22a) as

$$\begin{Bmatrix} dF_1 \\ dF_2 \\ dF_3 \end{Bmatrix} = \mathbf{K} \begin{Bmatrix} dl_x \\ dl_y \\ dl_z \end{Bmatrix} \quad \left(\mathbf{K} = \mathbf{F}^{-1} \right) \tag{22c}$$

were $\mathbf{K}$ is the nodal stiffness matrix.

The components of the flexibility matrix in Eq. (22b) are

$$\begin{aligned}
f_{11} &= \frac{\partial f}{\partial F_1} = -\frac{L_0}{EA_0} - \frac{1}{w}\left[\ln\left\{ F_3 + wL_0 + B \right\} - \ln\left\{ F_3 + A \right\} \right] \\
&\qquad - \frac{F_1^2}{w}\left[\frac{1}{B^2 + \left(F_3 + wL_0 \right) B} - \frac{1}{A^2 + F_3 A} \right] \\
f_{12} &= \frac{\partial f}{\partial F_2} = -\frac{F_1 F_2}{w}\left[\frac{1}{B^2 + \left(F_3 + wL_0 \right) B} - \frac{1}{A^2 + F_3 A} \right] \\
f_{13} &= \frac{\partial f}{\partial F_3} = -\frac{F_1}{w}\left[\frac{F_3 + wL_0 + B}{B^2 + \left(F_3 + wL_0 \right) B} - \frac{F_3 + A}{A^2 + F_3 A} \right] \\
f_{21} &= \frac{\partial g}{\partial F_1} = f_{12} \\
f_{22} &= \frac{\partial g}{\partial F_2} = -\frac{L_0}{EA_0} - \frac{1}{w}\left[\ln\left\{ F_3 + wL_0 + B \right\} - \ln\left\{ F_3 + A \right\} \right] \\
&\qquad - \frac{F_2^2}{w}\left[\frac{1}{B^2 + \left(F_3 + wL_0 \right) B} - \frac{1}{A^2 + F_3 A} \right] \\
f_{23} &= \frac{\partial g}{\partial F_3} = \frac{F_2}{F_1} f_{13} \\
f_{31} &= \frac{\partial h}{\partial F_1} = -\frac{F_1}{w}\left[\frac{1}{B} - \frac{1}{A} \right] \\
f_{32} &= \frac{\partial h}{\partial F_2} = \frac{F_2}{F_1} f_{31} \\
f_{33} &= \frac{\partial h}{\partial F_3} = -\frac{L_0}{EA_0} - \frac{1}{w}\left[\frac{F_3 + wL_0}{B} - \frac{F_3}{A} \right]
\end{aligned} \tag{23}$$

where,

$$A = \left(F_1^2 + F_2^2 + F_3^2 \right)^{1/2}, \; B = \left(F_1^2 + F_2^2 + \left(F_3 + wL_0 \right)^2 \right)^{1/2}$$

Thus, the tangent stiffness is obtained as follows:

$$\left\{ dF \right\} = \mathbf{K}_T \left\{ du \right\}$$

where

$$\mathbf{K}_T = \begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \frac{\partial F_1}{\partial u_3} & \frac{\partial F_1}{\partial u_4} & \frac{\partial F_1}{\partial u_5} & \frac{\partial F_1}{\partial u_6} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \frac{\partial F_2}{\partial u_3} & \frac{\partial F_2}{\partial u_4} & \frac{\partial F_2}{\partial u_5} & \frac{\partial F_2}{\partial u_6} \\ \frac{\partial F_3}{\partial u_1} & \frac{\partial F_3}{\partial u_2} & \frac{\partial F_3}{\partial u_3} & \frac{\partial F_3}{\partial u_4} & \frac{\partial F_3}{\partial u_5} & \frac{\partial F_3}{\partial u_6} \\ -\frac{\partial F_1}{\partial u_1} & -\frac{\partial F_1}{\partial u_2} & -\frac{\partial F_1}{\partial u_3} & -\frac{\partial F_1}{\partial u_4} & -\frac{\partial F_1}{\partial u_5} & -\frac{\partial F_1}{\partial u_6} \\ -\frac{\partial F_2}{\partial u_1} & -\frac{\partial F_2}{\partial u_2} & -\frac{\partial F_2}{\partial u_3} & -\frac{\partial F_2}{\partial u_4} & -\frac{\partial F_2}{\partial u_5} & -\frac{\partial F_2}{\partial u_6} \\ -\frac{\partial F_3}{\partial u_1} & -\frac{\partial F_3}{\partial u_2} & -\frac{\partial F_3}{\partial u_3} & -\frac{\partial F_3}{\partial u_4} & -\frac{\partial F_3}{\partial u_5} & -\frac{\partial F_3}{\partial u_6} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ij} \\ -\mathbf{K}_{ii} & -\mathbf{K}_{ij} \end{bmatrix} \tag{24}$$

However, since $dl_x = -du_1 = du_4, dl_y = -du_2 = du_5, dl_z = -du_3 = du_6$,

$$\mathbf{K}_{ii} = -\mathbf{K} \quad \text{and} \quad \mathbf{K}_{ij} = \mathbf{K} \tag{25}$$

and, therefore tangent stiffness matrix can be presented as

$$\mathbf{K}_T = -\begin{bmatrix} \mathbf{K} & -\mathbf{K} \\ -\mathbf{K} & \mathbf{K} \end{bmatrix} \tag{26}$$

It should be noted that when using the catenary cable element, along with section area and modulus of elasticity the weight density of the cable must be specified. This cable properties are used to define geometry of the cable and cable tension in initial configuration.