Tapered Beam Element

The Tapered Beam Element incorporated in the Finite Element Engine of MIDAS Family Programs is a three-dimensional beam element shown in Fig. 1. Formulation of this beam element is based on the following assumptions:

1. The element is a straight bar of uniform cross-section.
2. The beam cross-section is a closed solid (thick-walled) section
3. The element cross-section bending and shear centers are coincident.
4. The element is capable of resisting axial forces, bending moments about the two principal axes in the plane of its cross-section, and twisting moments about its centroidal axis.
5. The transverse shear effects are modeled according to Timoshenko beam theory.
6. Torsional behavior is governed by the Poison's theory of torsion (warping is not allowed).
7. At the element ends can be hinges (member end-releases).
8. At the element ends can be member end-offsets along its local axes.

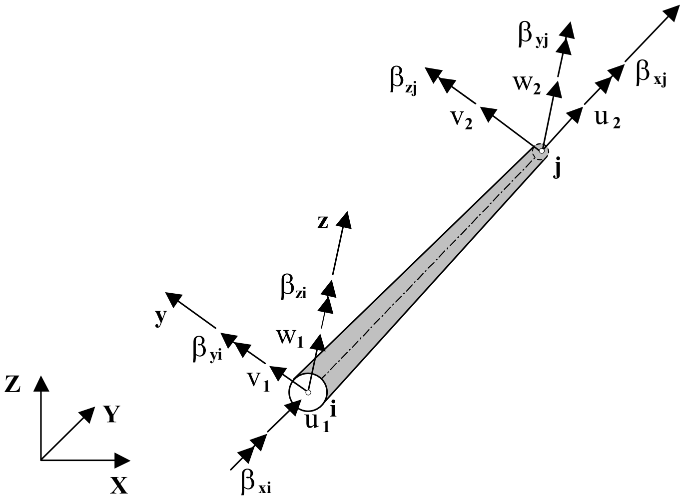

Figure 1 *A 3D Tapered Beam Finite Element*

The changes in cross section area, shear area, and torsional moment of inertia along the tapered beam element axis are assumed as follows:

$$A(x) = A_i + \left(A_j - A_i\right)\frac{x}{L}$$

$$A_s(x) = A_{s,i} + \left(A_{s,j} - A_{s,i}\right)\frac{x}{L}$$

$$I_{xx}(x) = I_{xx,i} + \left(I_{xx,j} - I_{xx,i}\right)\frac{x}{L}$$

Also, the change in moment of inertia along the length of a tapered beam is assumed as,

$$I(x) = \left(\sqrt[\alpha]{I_i} + \left(\sqrt[\alpha]{I_j} - \sqrt[\alpha]{I_i} \right) \frac{x}{L} \right)^{\alpha}$$

where, $0 < x < L$, and parameter α is

$$\alpha = \begin{cases} 1 & \text{- for linear} \\ 2 & \text{- for parabolic} \\ 3 & \text{- for cubic} \end{cases}.$$

According to the above equation, the stiffness varies more drastically along the axis with the increase in the value of α. In such a case, the overall stiffness of the member is smaller than that based on the linear variation.

Here, for the sake of convenience, the formulation has been formalized for the 2D beam element with uniform section (prismatic beam). The basic equilibrium equation for such element that defines the finite element stiffness matrix can be presented as follows:

$$\begin{bmatrix} k_{11} & 0 & 0 & k_{14} & 0 & 0 \\ & k_{22} & k_{23} & 0 & k_{25} & k_{26} \\ & & k_{33} & 0 & k_{35} & k_{36} \\ & & & k_{44} & 0 & 0 \\ & & & & k_{55} & k_{56} \\ \text{Sym.} & & & & & k_{66} \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ \theta_i \\ u_j \\ v_j \\ \theta_j \end{bmatrix} = \begin{bmatrix} H_i \\ V_i \\ M_i \\ H_j \\ V_j \\ M_j \end{bmatrix}$$

or in matrix form

$$\mathbf{Ku} = \mathbf{f}$$

where $\mathbf{K}$ is the stiffness matrix including terms that are defined using the standard finite element procedure based on displacement method. However, in the case of a tapered beam, the calculation of corresponding stiffness terms cannot be formalized and thus obtained by use of displacement method. So here we are using the force method, which can be summarized as follows. First, we assume that $H_i = 1$, and all the displacements except for u_i and the remaining loads at i-node are equal to 0, then the above equation becomes,

$$\begin{bmatrix} k_{11} & k_{14} \\ k_{14} & k_{44} \end{bmatrix} \begin{bmatrix} u_i \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Accordingly, $\mathbf{k}_{11}$ is identical to $1/u_i$. This displacement can be calculated by introducing a unit axial load while maintaining the j-end fixed as the axial direction based on the above equation of the assumed stiffness variation. The principle of virtual work, the method of consistent displacement, etc. can be used for calculations. If for example, the principle of virtual work is used, the equation then becomes,

$$u_i = \int_0^L \frac{1 \times 1}{EA(x)} dx$$

Following the above method and using the displacements calculated by introducing the unit shear force, unit moment, etc. at the i-end and j-end, we can back calculate the stiffness terms of the tapered beam element. First, if we apply a unit shear force at the i-end while maintaining the j-end fixed, we obtain,

$$\begin{bmatrix} k_{22} & k_{23} \\ k_{23} & k_{33} \end{bmatrix} \begin{bmatrix} v_{i1} \\ \theta_{i1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Similarly, introducing a unit moment at the i-end while maintaining the j-end fixed, the following is obtained:

$$\begin{bmatrix} k_{22} & k_{23} \\ k_{23} & k_{33} \end{bmatrix} \begin{bmatrix} v_{i2} \\ \theta_{i2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} k_{22} & k_{23} \\ k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} v_{i1} & v_{i2} \\ \theta_{i1} & \theta_{i2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} k_{22} & k_{23} \\ k_{32} & k_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_{i1} & v_{i2} \\ \theta_{i1} & \theta_{i2} \end{bmatrix}^{-1}$$

Accordingly, we obtain the four terms of the stiffness matrix. At this point, the displacements due to unit shear force and unit moment can be obtained by the following equations:

$$v_{i1} = \int_0^L \frac{V_1(x)^2}{EI(x)} dx + \int_0^L \frac{V_1(x)^2}{GA_s(x)} dx \qquad v_{i2} = \int_0^L \frac{V_2(x) V_1(x)}{EI(x)} dx$$

$$\theta_{i1} = \int_0^L \frac{M_1(x) M_2(x)}{EI(x)} dx \qquad \theta_{i2} = \int_0^L \frac{M_2(x)^2}{EI(x)} dx$$

where, $V_1(x)$ and $M_1(x)$ are the shear force and moment at the distance, x from the i-end due to the unit shear force applied at the i-end; and $V_2(x)$ and $M_2(x)$ are the shear force and moment at the distance, x from the i-end due to the unit moment applied at the i-end.

The remaining terms can be calculated by the same method noted above.